ST. ANNE'S COLLEGE OF ENGINEERING AND TECHNOLOGY
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MA3354- DISCRETE MATHEMATICS

UNIT-1

LOGIC AND PROOFS

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UNIT-1

LOGIC AND PROOFS

Propositional logic – Propositional equivalences - Predicates and quantifiers – Nested quantifiers –Rules of inference - Introduction to proofs – Proof methods and strategy.

UNIT-1 [LOGIC AND PROOFS]

Chapter - 1.1 [Propositional Logic]

Defn Propositions

A proposition (statement) is a declarative sentence that is either true or false, but not both.

Eg:

i) $2+6=8$ (T) ; (ii) $3+5=10$ (F)

iii) Panruti is in Delhi (F)

iv) Chennai is a capital of Tamilnadu (T)

Five Basic Connectives

English sentence	Logical connectives	symbols
and	Conjunction	$\wedge$ (meet)
or	Disjunction	$\vee$ (Joint)
not	Negation	$\neg$ (or) $\sim$
if... then	Conditional	$\rightarrow$
if and only if	Biconditional	$\leftrightarrow$

Conjunction (or) AND ($\wedge$) (meet)

The Conjunction of the two statements P, Q is given by $P \wedge Q$ (read as P and Q)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (or) OR ($\vee$) (Joint)

The disjunction of the two Statements P, Q is given by $P \vee Q$ (read as P or Q)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation (or) NOT ($\sim$, $\neg$)

Let P be the statement, then the "negation of P" is defined by $\neg P$ (or) $\sim P$ (read as "not P")

P	$\neg P$ (or) $\sim P$
T	F
F	T

Conditional ($\rightarrow$)

Let P, Q are the two statements, then the statements $P \rightarrow Q$ (read as if P, then Q) is called conditional statement.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional ($\leftrightarrow$)

Let P, Q are two statements, then the statements $P \leftrightarrow Q$ (read as P if and only if Q) (or) (P iff Q) is called biconditional statement.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Converse:-

Let $P \to Q$ be the statement. then the Converse of $P \to Q$ is given by $Q \to P$

Contrapositive:-

Let $P \to Q$ be the statement, then the Contrapositive of $P \to Q$ is given by $\neg Q \to \neg P$

Inverse:-

Let $P \to Q$ be the statement. then the inverse of $P \to Q$ is given by $\neg P \to \neg Q$

Example-①

Construct the truth table for the following statement (i) $\neg P \vee \neg Q$, (ii) $\neg P \vee Q$, (iii) $\neg Q \to \neg P$ (iv) $\neg P \to \neg Q$ (v) $Q \to P$. (vi) $\neg Q \wedge P$

Soln:

(i) $\neg P \vee \neg Q$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(ii) $\neg P \vee Q$

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(iii) $\neg Q \to \neg P$ (iv) $\neg P \to \neg Q$

P	Q	$\neg P$	$\neg Q$	$\neg Q \to \neg P$	$\neg P \to \neg Q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

(v) $Q \to P$

P	Q	$Q \to P$
T	T	T
T	F	T
F	T	F
F	F	T

(vi) $\neg P \vee Q$

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Example-②

Construct the truth table for the following statement

(i) $P \wedge (P \vee Q)$, (ii) $(P \vee Q) \wedge Q$, (iii) $(P \to Q) \wedge (Q \to P)$

(iv) $(P \to Q) \vee Q$

Soln) (i) $P \wedge (P \vee Q)$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(ii) $(P \vee Q) \wedge Q$

P	Q	$P \vee Q$	$(P \vee Q) \wedge Q$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	F

(iii) $(P \to Q) \wedge (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(iv) $(P \to Q) \vee Q$

P	Q	$(P \to Q)$	$(P \to Q) \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Example - (3) Construct the truth table for the following

(i) $P \wedge \neg P$, (ii) $P \vee \neg P$, (iii) $P \wedge P$, (iv) $P \vee P$

Soln

(i) $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(ii) $P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

(iii) $P \wedge P$

P	P	$P \wedge P$
T	T	T
F	F	F

(iv) $P \vee P$

P	P	$P \vee P$
T	T	T
F	F	F

Example - (4) Construct the truth table for $\neg(P \vee Q) \wedge (P \vee R)$

Sol^n:

Given that $\neg(P \vee Q) \wedge (P \vee R)$

P	Q	R	$(P \vee Q)$	$\neg(P \vee Q)$	$(P \vee R)$	$\neg(P \vee Q) \wedge (P \vee R)$
T	T	T	T	F	T	F
T	T	F	T	F	T	F
T	F	T	T	F	T	F
T	F	F	T	F	T	F
F	T	T	T	F	T	F
F	T	F	T	F	F	F
F	F	T	F	T	T	T
F	F	F	F	T	F	F

Example (5)

Construct the truth table for $\neg(P \wedge Q) \longleftrightarrow (\neg P \vee \neg Q)$

Sol^n:

Let $S : \neg(P \wedge Q) \longleftrightarrow (\neg P \vee \neg Q)$

Here, $a = \neg(P \wedge Q)$ & $b = (\neg P \vee \neg Q)$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P \vee \neg Q)$	$a \longleftrightarrow b$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Example -(6) Construct the truth table for the following table

$[\sim P \wedge (\sim Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)]$

Soln: $a : \sim P \wedge (\sim Q \wedge R)$ & $b : (Q \wedge R) \vee (P \wedge R)$

$S : [\sim P \wedge (\sim Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)]$

P	Q	R	~P	~Q	(~Q∧R)	a	(Q∧R)	(P∧R)	b	S: a∨b
T	T	T	F	F	F	F	T	T	T	T
T	T	F	F	F	F	F	F	F	F	F
T	F	T	F	T	T	F	F	T	T	T
T	F	F	F	T	F	F	F	F	F	F
F	T	T	T	F	F	F	T	F	T	T
F	T	F	T	F	F	F	F	F	F	F
F	F	T	T	T	T	T	F	F	F	T
F	F	F	T	T	F	F	F	F	F	F

Example -(7) Construct the truth table

$\neg[P \vee (Q \wedge R)] \longleftrightarrow [(P \vee Q) \wedge (P \vee R)]$

Soln: $S : \neg[P \vee (Q \wedge R)] \longleftrightarrow [(P \vee Q) \wedge (P \vee R)]$

Here, $a : \neg[P \vee (Q \wedge R)]$ & $b : [(P \vee Q) \wedge (P \vee R)]$

P	Q	R	Q∧R	P∨Q	P∨R	P∨(Q∧R)	a	b	a↔b
T	T	T	T	T	T	T	F	T	F
T	T	F	F	T	T	T	F	T	F
T	F	T	F	T	T	T	F	T	F
T	F	F	F	T	T	T	F	T	F
F	T	T	T	T	T	T	F	T	F
F	T	F	F	T	F	F	T	F	F
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	F	F	T	F	F

Example-(8)

Write the following statement in symbolic form.

If Either Ram takes calculus or krishna takes sociology then sita will takes English.

Soln:

Let a: Ram takes calculus

b: krishna takes sociology

c: sita will take English

symbol form: $(a \vee b) \longrightarrow c$

Example-(9)

How can this English sentence be translated into a logical expression?

" You can access the internet from campus only If you are computer science major or you are not a fresh man.

Soln:

a: you can access the internet from campus

b: You are a computer science major

c: You are a fresh man

logical expression: $a \longrightarrow (b \vee \neg c)$

Example-(10)

Let P and Q be the propositions " swimming at the Chennai shore is allowed" and "sharks have be spotted near the shore" respectively.

Express Each of these Compound Propositions as an English sentence. (i) $\neg Q$ (ii) $\neg P \vee Q$ (iii) $P \wedge Q$ (iv) $\neg Q \to P$
(v) $P \to \neg Q$ (vi) $\neg P \wedge (P \vee \neg Q)$

Soln: Let P: swimming at the Chennai shore is allowed
Q: Sharks have been spotted near the shore.

(i) $\neg Q$: sharks have not been spotted near the shore

(ii) $\neg P \vee Q$: swimming at the Chennai shore is not allowed or sharks have been spotted near the shore.

(iii) $P \wedge Q$: swimming at the Chennai shore is allowed and sharks have been spotted near the shore.

iv) $\neg Q \to P$: If sharks have ~~been~~ not been spotted near the shore, then swimming at the Chennai shore is allowed

v) $P \to \neg Q$: If swimming at the chennai shore is allowed, then sharks have not been spotted near the shore.

vi) $\neg P \wedge (P \vee \neg Q)$: swimming at the Chennai shore is not allowed and Either swimming at the Chennai shore is allowed or sharks have not been spotted near the shore.

Logical Equivalances

Tautology: A statement that is true for all possible proposition variables is called Tautology.

Contradiction: A statement that is always False is called Contradiction.

Eg-① Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

Solu

Let $S : [Q \vee (P \wedge \sim Q)] \vee (\sim P \wedge \sim Q)$

P	Q	$\sim P$	$\sim Q$	$P \wedge \sim Q$	$Q \vee (P \wedge \sim Q)$	$(\sim P \wedge \sim Q)$	S
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

$\therefore$ The statement S is tautology.

Eg-② Show that $\neg P \rightarrow (P \rightarrow Q)$ is a tautology

Solu

Let $\neg P \rightarrow (P \rightarrow Q)$

P	Q	$\neg P$	$(P \rightarrow Q)$	$\neg P \rightarrow (P \rightarrow Q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

The given statement is tautology

Eg-③ Using the truth table verify that the Proposition $(P \wedge Q) \wedge \neg(P \vee Q)$

Solu:

Let $S : (P \wedge Q) \wedge \neg(P \vee Q)$

P	Q	$P \wedge Q$	$P \vee Q$	$\neg(P \vee Q)$	$S : (P \wedge Q) \wedge \neg(P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The given statement is contradiction

Logically Equivalent:

The two statement P and Q said to be logically Equivalent. If $P \Leftrightarrow Q$ is a tautology.

Eg-④ Show that P is Equivalent to the following formula

(i) $P \Leftrightarrow \neg\neg P$ (ii) $P \Leftrightarrow P \wedge P$, (iii) $P \Leftrightarrow P \vee (P \wedge Q)$, (iv) $P \Leftrightarrow P \wedge (P \vee Q)$

Soln.

1	2	3	4	5	6	7	8	9
P	Q	$\neg P$	$\neg\neg P$	$P \wedge P$	$P \wedge Q$	$P \vee Q$	$P \vee (P \wedge Q)$	$P \wedge (P \vee Q)$
T	T	F	T	T	T	T	T	T
T	F	F	T	T	F	T	T	T
F	T	T	F	F	F	T	F	F
F	F	T	F	F	F	F	F	F

(i) Column ① & ④ are Equal, $P \Leftrightarrow \neg\neg P$

(ii) Column ① & ⑤ are Equal, $P \Leftrightarrow P \wedge P$

(iii) Column ① & ⑧ are Equal, $P \Leftrightarrow P \vee (P \wedge Q)$

(iv) Column ① & ⑨ are Equal, $P \Leftrightarrow P \wedge (P \vee Q)$

eg-⑤ Show the following Equivalances

(i) $(P \rightarrow Q) \Leftrightarrow \neg P \vee Q$ (ii) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

(iii) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Soln:

1	2	3	4	5	6	7	8	9	10
P	Q	$\neg P$	$\neg Q$	$(P \rightarrow Q)$	$(\neg P \vee Q)$	$(\neg P \vee \neg Q)$	$(\neg P \wedge \neg Q)$	$(P \wedge Q)$	$\neg(P \wedge Q)$
T	T	F	F	T	T	F	F	T	F
T	F	F	T	F	F	T	F	F	T
F	T	T	F	T	T	T	F	F	T
F	F	T	T	T	T	T	T	F	T

11	12
$(P \vee Q)$	$\neg(P \vee Q)$
T	F
T	F
T	F
F	T

(i) Column 5 and 6 are Equal

$(P \rightarrow Q) \Leftrightarrow \neg P \vee Q$

(ii) Column 10 and 7 are Equal

$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

(iii) Column 12 and 8 are Equal

$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Eg-(6) Show the following Equivalances

(i) $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

(ii) $\neg(P \leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Soln

(i) $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

1	2	3	4	5	6
P	Q	$\neg Q$	$P \rightarrow Q$	$\neg(P \rightarrow Q)$	$P \wedge \neg Q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(ii) $\neg(P \leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

1	2	3	4	5	6	7	8	9
P	Q	$\neg P$	$\neg Q$	$(P \leftrightarrow Q)$	$\neg(P \leftrightarrow Q)$	$(P \wedge \neg Q)$	$(\neg P \wedge Q)$	$(P \wedge \neg Q) \vee (\neg P \wedge Q)$
T	T	F	F	T	F	F	F	F
T	F	F	T	F	T	T	F	T
F	T	T	F	F	T	F	T	T
F	F	T	T	T	F	F	F	F

(i) Column (5) & (6) are Equal

$\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

(ii) Column (6) & (9) are Equal

$\neg(P \leftrightarrow Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Eg-7 Show that P is Equivalances to $P \wedge (P \vee Q)$ and $(P \vee Q) \wedge (P \vee \neg Q)$

Soln:

Given that (i) $P \Leftrightarrow P \wedge (P \vee Q)$; (ii) $P \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q)$

(i) Truth table for $P \Leftrightarrow P \wedge (P \vee Q)$

1	2	3	4
P	Q	$P \vee Q$	$P \wedge (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Column (1) & (4) are Equal

$P \Leftrightarrow P \wedge (P \vee Q)$

(ii) Truth table for $P \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q)$

1	2	3	4	5	6
P	Q	$\neg Q$	$(P \vee Q)$	$(P \vee \neg Q)$	$(P \vee Q) \wedge (P \vee \neg Q)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

Column (1) & (6) are Equal

$P \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q)$

Chapter-1.2 Logic Equivalance Table

Equivalance	Name
$P \wedge T \Longleftrightarrow P$ $P \vee F \Longleftrightarrow P$	Identity Law
$P \vee T \Longleftrightarrow T$ $P \wedge F \Longleftrightarrow F$	Domination Law
$P \vee \neg P \Longleftrightarrow T$ $P \wedge \neg P \Longleftrightarrow F$	Negation Law
$P \vee Q \Longleftrightarrow Q \vee P$ $P \wedge Q \Longleftrightarrow Q \wedge P$	Commutative Law
$P \vee P \Longleftrightarrow P$ $P \wedge P \Longleftrightarrow P$	Idempotent Law
$\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$ $\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$	De margon's Law
$P \wedge (P \vee Q) \Longleftrightarrow P$ $P \vee (P \wedge Q) \Longleftrightarrow P$	Absorption Law
$\neg\neg P \Longleftrightarrow P$	Double Negation Law
$P \vee (Q \vee R) \Longleftrightarrow (P \vee Q) \vee R$ $P \wedge (Q \wedge R) \Longleftrightarrow (P \wedge Q) \wedge R$	Associative Law
$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$ $(P \vee Q) \wedge R \Longleftrightarrow (P \wedge R) \vee (Q \wedge R)$ $(P \wedge Q) \vee R \Longleftrightarrow (P \vee R) \wedge (Q \vee R)$	Distributive Law

Important Equivalance

1. $P \rightarrow Q \Longleftrightarrow \neg P \vee Q$
2. $\neg(P \rightarrow Q) \Longleftrightarrow P \wedge \neg Q$
3. $P \rightarrow Q \Longleftrightarrow \neg Q \rightarrow \neg P$
4. $P \leftrightarrow Q \Longleftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$
5. $P \leftrightarrow Q \Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$
6. $\neg(P \leftrightarrow Q) \Longleftrightarrow P \leftrightarrow \neg Q$

Eg-① Show that $(P \vee Q) \wedge \neg(\neg P \wedge Q) \Leftrightarrow P$

Sol'n: Given that $(P \vee Q) \wedge \neg(\neg P \wedge Q) \Leftrightarrow P$

L.H.S $(P \vee Q) \wedge \neg(\neg P \wedge Q)$	Reasons
$\Leftrightarrow (P \vee Q) \wedge (\neg\neg P \vee \neg Q)$	De Morgan's Law
$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q)$	Double Negation Law
$\Leftrightarrow P \vee (Q \wedge \neg Q)$	Distributive Law
$\Leftrightarrow P \vee F$	Negation Law
$\Leftrightarrow P$ (R.H.S)	Identity Law

Eg-② Show that $\neg[\neg((P \vee Q) \wedge R) \vee \neg Q] \Leftrightarrow Q \wedge R$

Sol'n: Given that $\neg[\neg((P \vee Q) \wedge R) \vee \neg Q] \Leftrightarrow Q \wedge R$

L.H.S $\neg[\neg((P \vee Q) \wedge R) \vee \neg Q]$	Reasons
$\Leftrightarrow \neg\neg[(P \vee Q) \wedge R] \wedge \neg\neg Q$	De Morgan's law
$\Leftrightarrow [(P \vee Q) \wedge R] \wedge Q$	Double Negation Law
$\Leftrightarrow (P \vee Q) \wedge (R \wedge Q)$	Associative Law
$\Leftrightarrow (P \vee Q) \wedge (Q \wedge R)$	Commutative Law
$\Leftrightarrow [(P \vee Q) \wedge Q] \wedge R$	Associative Law
$\Leftrightarrow Q \wedge R$ (R.H.S)	Absorption Law

Eg-(3) Show that $[\neg P \wedge (\neg Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)] \Leftrightarrow R$

Soln

Given that $[\neg P \wedge (\neg Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)] \Leftrightarrow R$

(i) $\neg P \wedge (\neg Q \wedge R)$	
$\Leftrightarrow (\neg P \wedge \neg Q) \wedge R$	Associative Law
$\Leftrightarrow \neg (P \vee Q) \wedge R$	De morgan Law
(ii) $(Q \wedge R) \vee (P \wedge R)$	
$\Leftrightarrow (Q \vee P) \wedge R$	Distributive Law
$\Leftrightarrow (P \vee Q) \wedge R$	Commutative Law
(L.H.S) $[\neg P \wedge (\neg Q \wedge R)] \vee [(Q \wedge R) \vee (P \wedge R)]$	
$\Leftrightarrow [\neg (P \vee Q) \wedge R] \vee [(P \vee Q) \wedge R]$	using (i) & (ii)
$\Leftrightarrow [\neg (P \vee Q) \vee (P \vee Q)] \wedge R$	Distributive Law
$\Leftrightarrow T \vee R$	Negation Law
$\Leftrightarrow R$ (R.H.S)	Identity Law

Eg-(4) Show that $(P \vee Q) \wedge \neg P \Leftrightarrow \neg P \wedge Q$.

Soln

Given that $(P \vee Q) \wedge \neg P \Leftrightarrow \neg P \wedge Q$

R.H.S $(P \vee Q) \wedge \neg P$	Reasons
$\Leftrightarrow \neg P \wedge (P \vee Q)$	Commutative Law
$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge Q)$	Distributive Law
$\Leftrightarrow F \vee (\neg P \wedge Q)$	Negation Law
$\Leftrightarrow \neg P \wedge Q$	Identity Law

Eg ⑤ Show that $\neg[P \vee (\neg P \wedge Q)]$ and $\neg P \wedge \neg Q$ are logically Equivalent.

Soln: Given that $\neg[P \vee (\neg P \wedge Q)] \Leftrightarrow \neg P \wedge \neg Q$

R-H-S $\neg[P \vee (\neg P \wedge Q)]$	Reasons
$\Leftrightarrow \neg P \wedge \neg(\neg P \wedge Q)]$	De morgan's Law
$\Leftrightarrow \neg P \wedge [\neg(\neg P) \vee \neg Q]$	De morgan's Law
$\Leftrightarrow \neg P \wedge [P \vee \neg Q]$	Double Negation law
$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$	Distribative law
$\Leftrightarrow F \vee (\neg P \wedge \neg Q)$	Negation Law $(P \wedge \neg P) \Leftrightarrow F$
$\Leftrightarrow (\neg P \wedge \neg Q) \vee F$	Commutative Law
$\Leftrightarrow \neg P \wedge \neg Q$ (R-H-S)	$P \vee F \Leftrightarrow P$

Eg-⑥ Show that $(P \vee Q) \wedge [\neg P \wedge (\neg P \wedge Q)] \Leftrightarrow (\neg P \wedge Q)$

Soln:- Given that $(P \vee Q) \wedge [\neg P \wedge (\neg P \wedge Q)] \Leftrightarrow (\neg P \wedge Q)$

L-H-S $(P \vee Q) \wedge [\neg P \wedge (\neg P \wedge Q)]$	Reasons
$\Leftrightarrow (P \vee Q) \wedge [(\neg P \wedge \neg P) \wedge Q]$	Associative Law
$\Leftrightarrow (P \vee Q) \wedge [\neg P \wedge Q]$	Since, $\neg P \wedge \neg P \Leftrightarrow \neg P$
$\Leftrightarrow [(P \vee Q) \wedge \neg P] \wedge Q$	Associative Law
$\Leftrightarrow [(P \wedge \neg P) \vee (Q \wedge \neg P)] \wedge Q$	Distributive Law
$\Leftrightarrow [F \vee (Q \wedge \neg P)] \wedge Q$	Negation Law
$\Leftrightarrow (Q \wedge \neg P) \wedge Q$	Identity Law
$\Leftrightarrow (\neg P \wedge Q) \wedge Q$	Commatative Law
$\Leftrightarrow \neg P \wedge (Q \wedge Q)$	Associative Law
$\Leftrightarrow \neg P \wedge Q$ (R-H-S)	Idempotent Law

Eg-7 Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

Soln: Given that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology

ie, $(P \wedge Q) \rightarrow (P \vee Q) \Leftrightarrow T$

L.H.S $(P \wedge Q) \rightarrow (P \vee Q)$	Reasons
$\Leftrightarrow \neg(P \wedge Q) \vee (P \vee Q)$	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$	De morgan's Law
$\Leftrightarrow (\neg Q \vee \neg P) \vee (P \vee Q)$	Commutative Law
$\Leftrightarrow \neg Q \vee [(\neg P \vee P) \vee Q)]$	Associative Law
$\Leftrightarrow (\neg Q \vee T) \vee Q$	Negation law
$\Leftrightarrow \neg Q \vee Q$	Domination Law
$\Leftrightarrow T$ (R.H.S)	Negation. Law

Eg-8 Show that $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R \Leftrightarrow P \rightarrow (\neg Q \vee R)$

Soln: Given that $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R \Leftrightarrow P \rightarrow (\neg Q \vee R)$

(i) $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

R.H.S $P \rightarrow (Q \rightarrow R)$	Reasons
$\Leftrightarrow P \rightarrow (\neg Q \vee R)$	$a \rightarrow b \Leftrightarrow \neg a \vee b$
$\Leftrightarrow \neg P \cancel{\vee} (\neg Q \vee R)$	$a \rightarrow b \Leftrightarrow \neg a \vee b$
$\Leftrightarrow (\neg P \vee \neg Q) \vee R$	Associative Law
$\Leftrightarrow \neg(P \wedge Q) \vee R$	De margon's Law
$\Leftrightarrow (P \wedge Q) \rightarrow R$ (R.H.S)	$\neg a \vee b \Leftrightarrow a \rightarrow b$

(ii) $(P \wedge Q) \to R \iff P \to (\neg Q \vee R)$

L.H.S $(P \wedge Q) \to R$	Reasons
$\iff \neg(P \wedge Q) \vee R$	$a \to b \iff \neg a \vee b$
$\iff (\neg P \vee \neg Q) \vee R$	De morgans Law
$\iff \neg P \vee (\neg Q \vee R)$	Associative Law
$\iff P \to (\neg Q \vee R)$ (R.H.S)	$\neg a \vee R \to a \to b$

from ① & ② $P \to (Q \to R) \iff (P \wedge Q) \to R \iff P \to (\neg Q \vee R)$ is Proved

Eg-⑨

Show that $[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee [(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)]$ is tautology.

Soln

Given that $S: [(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee [(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)]$

(i) $(P \vee Q) \wedge \neg[\neg P \wedge (\neg Q \vee \neg R)]$	Reasons
$\iff (P \vee Q) \wedge \neg[\neg P \wedge \neg(Q \wedge R)]$	De morgan's Law
$\iff (P \vee Q) \wedge [\neg\neg P \vee \neg\neg(Q \wedge R)]$	De morgan's Law
$\iff (P \vee Q) \wedge [P \vee (Q \wedge R)]$	Double negation Law
$\iff P \vee [Q \wedge (Q \wedge R)]$	Distributive Law
$\iff P \vee [(Q \wedge Q) \wedge R]$	Associative law
$\iff P \vee (Q \wedge R)$	Idempotent Law

(ii) $(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$	Reasons
$\Leftrightarrow \neg(P \vee Q) \vee \neg(P \vee R)$	De-morgan's Law
$\Leftrightarrow \neg[(P \vee Q) \wedge (P \vee R)]$	De morgan's Law
$\Leftrightarrow \neg[P \vee (Q \wedge R)]$	Distributive Law

from (1) & (2)

$S : [(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee [(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)]$

$\Leftrightarrow [P \vee (Q \wedge R)] \vee \neg[P \vee (Q \wedge R)]$

$\Leftrightarrow Q \vee \neg Q$ $\qquad\qquad \{P \vee \neg P \Leftrightarrow T\}$

$\Leftrightarrow T$

$\therefore$ S is Tautology

Chapter - 1.3

PDNF and PCNF

Defn: Normal Form (or) Canonical Form

If we write the given statement formula in a particular form ie, interms of $\wedge, \vee, \sim$ then it is called normal form.

Defn: Elementary Product

A product of the variables and their negations in a formula is called an elementary product (product means conjunction ie, $\wedge$)

Eg: Let P and Q be any two atomic variables. Then possible elementary product are $P \wedge \neg Q$, $\neg P \wedge Q$, $P \wedge \neg P$, $Q \wedge \neg Q$.

Defn: Elementary Sum

A sum of the variables and their negations in a formula is called an elementary sum (sum means disjunction ie, $\vee$)

Eg: Let P and Q be any two atomic variables then possible elementary sum are $P \vee Q$, $\neg P \vee Q$, $P \vee \neg Q$, $P \vee \neg P$

Defn: DNF – Disjunctive Normal Form.

A formula which is Equivalent to a given formula and which consists of a sum of Elementary Products is called a Disjunctive Normal Form (DNF)

Defn: CNF – Conjunctive Normal Form

A formula which is Equivalent to a given formula and which consists of a sum of Elementary sum is called Conjunctive Normal Form (CNF)

Eg:

$$DNF = \begin{pmatrix} \text{Elementary} \\ \text{Product} \end{pmatrix} \vee \begin{pmatrix} \text{Elementary} \\ \text{Product} \end{pmatrix} \vee (\ldots \ldots \vee \begin{pmatrix} \text{Elementary} \\ \text{Product} \end{pmatrix}$$

$$CNF = \begin{pmatrix} \text{Elementary} \\ \text{Sum} \end{pmatrix} \wedge \begin{pmatrix} \text{Elementary} \\ \text{Sum} \end{pmatrix} \wedge \ldots \ldots \wedge \begin{pmatrix} \text{Elementary} \\ \text{Sum} \end{pmatrix}$$

Defn: Min terms

1) Let P and Q be two statement variables, then the min terms are $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$

2) Let P, Q and R be three statements variables, then the min terms are, $P \wedge Q \wedge R$, $\neg P \wedge Q \wedge R$, $P \wedge \neg Q \wedge R$, $P \wedge Q \wedge \neg R$, ..

Defn: Max terms

1) Let P and Q be two statement variables, then the max terms are, $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$

2) Let P, Q and R be three statement variable, then the max terms are, $P \vee Q \vee R$, $\neg P \vee Q \vee R$, $P \vee \neg Q \vee R$, $P \vee Q \vee \neg R$

Def^n: Principal Disjunctive Normal Form (PDNF)

A formula which is Equivalent to a given formula and which consists of sum of its min terms is called PDNF.

Def^n: Principal Conjunctive Normal Form (PCNF)

A formula which is Equivalent to a given formula and which consists of product of its max terms is called PCNF.

Eg:-

PDNF = (min terms) $\vee$ (min terms) $\vee$ - - - - $\vee$ (min terms)

PCNF = (max terms) $\wedge$ (max terms) $\wedge$ - - - - $\wedge$ (max terms)

Eg-① obtain a disjunctive normal form of

$P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$.

solⁿ

Given that

$P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$

$\Leftrightarrow P \to [(\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P)]$ {using $P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow P \to [(\neg P \vee Q) \wedge (\neg\neg Q \wedge \neg\neg P)]$ {De-morgan's law

$\Leftrightarrow P \to [(\neg P \vee Q) \wedge (Q \wedge P)]$ {Double Negation law

$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$ $\quad P \to Q \Leftrightarrow \neg P \vee Q$

$\Longleftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$

$\Longleftrightarrow \neg P \vee [(\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))]$

$\Longleftrightarrow \neg P \vee [(\neg P \wedge (P \wedge Q)) \vee ((Q \wedge Q) \wedge P)]$

$\Longleftrightarrow \neg P \vee [((\neg P \wedge P) \wedge Q) \vee (Q \wedge P)]$

$\Longleftrightarrow \neg P \vee [(F \wedge Q) \vee (Q \wedge P)]$

$\Longleftrightarrow \neg P \vee [F \vee (Q \wedge P)]$

$\Longleftrightarrow \neg P \vee (Q \wedge P)$ is DNF

{Distribution law
$(P \vee Q) \wedge R = (P \wedge R) \vee (Q \wedge R)$

Commutative law
$P \wedge Q = Q \wedge P$

Associative law
$(P \vee Q) \vee R = P \vee (Q \vee R)$

$Q \wedge Q = Q$ & $\neg P \wedge P = F$

Domination law
$F \wedge P = F$

Identity $F \vee P = P$

Eg-② obtain a disjunctive normal form of

$(P \wedge \neg(Q \vee R)) \vee [((P \wedge Q) \vee \neg R) \wedge P]$

Soln:

$[P \wedge \neg(Q \vee R)] \vee [((P \wedge Q) \vee \neg R) \wedge P]$

$\Longleftrightarrow [P \wedge (\neg Q \wedge \neg R)] \vee [((P \wedge Q) \vee \neg R) \wedge P]$ {De Morgan's Law

$\Longleftrightarrow [P \wedge \neg Q \wedge \neg R] \vee [((P \wedge Q) \wedge P) \vee (\neg R \wedge P)]$

Distributive Law

$\Longleftrightarrow [P \wedge \neg Q \wedge \neg R] \vee [(P \wedge (Q \wedge P)) \vee (\neg R \wedge P)]$

Associative Law

$\Longleftrightarrow [P \wedge \neg Q \wedge \neg R] \vee [P \wedge (P \wedge Q)) \vee (\neg R \wedge P)]$ Commutative Law

$\Longleftrightarrow [P \wedge \neg Q \wedge \neg R] \vee [(P \wedge P) \wedge Q) \vee (\neg R \wedge P)]$ Associative Law

$\Longleftrightarrow [P \wedge \neg Q \wedge \neg R] \vee (P \wedge Q) \vee (\neg R \wedge P)$ Idempotent Law

This is required DNF.

③ obtain a Conjunctive normal form for
$[P \to (Q \wedge R)] \wedge [\neg P \to (\neg Q \wedge \neg R)]$

Soln:-

$[P \to (Q \wedge R)] \wedge [\neg P \to (\neg Q \wedge \neg R)]$

$\Leftrightarrow [\neg P \vee (Q \wedge R)] \wedge [P \vee (\neg Q \wedge \neg R)]$ $\{P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R) \wedge (P \vee \neg Q) \wedge (P \vee \neg R)$ Distributive Law

This is Required CNF.

④ obtain a Conjunctive normal form of the formula
$P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$

Soln:-

$P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$

$\Leftrightarrow \neg P \vee [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$ $\{P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P)]$ $\{P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$ $\{$De morgan's Law

$\Leftrightarrow [\neg P \vee (\neg P \vee Q)] \wedge [\neg P \vee (Q \wedge P)]$ Distributive Law

$\Leftrightarrow [(\neg P \vee \neg P) \vee Q] \wedge [(\neg P \vee Q) \wedge (\neg P \vee P)]$ Associative Law

$\Leftrightarrow (\neg P \vee Q) \wedge [(\neg P \vee Q) \wedge T]$ Negation Law

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee Q)$ Identity Law

This is a Required CNF.

To find PDNF and PCNF

① obtain the Principal disjunctive normal form (PDNF) $\neg P \vee Q$ (or) $P \to Q$. Also find PCNF

Soln: Let $S \Leftrightarrow \neg P \vee Q$

P	Q	$\neg P$	$\neg P \vee Q$	minterms	maxterms
T	T	F	T	$P \wedge Q$	
T	F	F	F		$\neg P \vee Q$
F	T	T	T	$\neg P \wedge Q$	
F	F	T	T	$\neg P \wedge \neg Q$	

$\therefore$ $S \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is PDNF

& $S \Leftrightarrow \neg P \vee Q$ is PCNF.

② obtain PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ Also find PCNF

Soln: Let $S \Leftrightarrow (P \wedge Q) \vee [(\neg P \wedge R) \vee (Q \wedge R)]$

Here $A : (\neg P \wedge R) \vee (Q \wedge R)$ & $B = P \wedge Q$

P	Q	R	$\neg P$	$B = P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	A	$S = B \vee A$	minterm	maxterms
T	T	T	F	T	F	T	T	T	$P \wedge Q \wedge R$	
T	T	F	F	T	F	F	F	T	$P \wedge Q \wedge \neg R$	
T	F	T	F	F	F	F	F	F		$\neg P \vee Q \vee \neg R$
T	F	F	F	F	F	F	F	F		$\neg P \vee Q \vee R$
F	T	T	T	F	T	T	T	T	$\neg P \wedge Q \wedge R$	
F	T	F	T	F	F	F	F	F		$P \vee \neg Q \vee R$
F	F	T	T	F	T	F	T	T	$\neg P \wedge \neg Q \wedge R$	
F	F	F	T	F	F	F	F	F		$P \vee Q \vee R$

The PDNF is

$S \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

The PCNF is

$S \Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \vee (P \vee \neg Q \vee R) \vee (P \vee Q \vee R)$

(3) obtain the Principal disjunctive and conjuctive normal forms $[P \rightarrow (Q \wedge R)] \wedge [\neg P \rightarrow (\neg Q \wedge \neg R)]$

To find PCNF and PDNF.

Soln:

Let $S \Leftrightarrow [P \rightarrow (Q \wedge R)] \wedge [\neg P \rightarrow (\neg Q \wedge \neg R)]$

Here, $A = P \rightarrow (Q \wedge R)$; $B = \neg Q \wedge \neg R$; $C = \neg P \rightarrow B$

$\therefore S = A \wedge C$

P	Q	R	¬P	¬Q	¬R	Q∧R	B= ¬Q∧¬R	A	C	S=A∧C	minterms	maxterms
T	T	T	F	F	F	T	F	T	T	T	P∧Q∧R	
T	T	F	F	F	T	F	F	F	T	F		¬P∨¬Q∨R
T	F	T	F	T	F	F	F	F	T	F		¬P∨Q∨¬R
T	F	F	F	T	T	F	T	F	T	F		¬P∨Q∨R
F	T	T	T	F	F	T	F	T	F	F		P∨¬Q∨¬R
F	T	F	T	F	T	F	F	T	F	F		P∨¬Q∨R
F	F	T	T	T	F	F	F	T	F	F		P∨Q∨¬R
F	F	F	T	T	T	F	T	T	T	T	¬P∧¬Q∧¬R	

The PDNF $= (P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

The PCNF $= (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$
$\wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R)$

(4) Without constructing the truth table obtain the PDNF and PCNF of the formula. $(\neg P \to R) \wedge (Q \leftrightarrow P)$.

Sol^n:

Let $S \Leftrightarrow (\neg P \to R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \to P) \wedge (P \to Q)]$ $\left\{ \begin{array}{l} P \to Q \Leftrightarrow \neg P \vee Q \\ Q \leftrightarrow P \Leftrightarrow (Q \to P) \wedge (P \to Q) \end{array} \right.$

$\Leftrightarrow (P \vee R) \wedge [(\neg Q \vee P) \wedge (\neg P \vee Q)]$ $\left\{ \begin{array}{l} P \to Q \Leftrightarrow \neg P \vee Q \\ \text{Double negation Law} \end{array} \right.$

$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$ $\{ P \vee F \Leftrightarrow P$

$\Leftrightarrow [(P \vee R) \vee F] \wedge [(\neg Q \vee P) \vee F] \wedge [(\neg P \vee Q) \vee F]$ $\{ P \wedge \neg P \Leftrightarrow F$

$\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [(\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$

$\Leftrightarrow [(P \vee R \vee Q) \wedge (P \vee R \vee \neg Q)] \wedge [(\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R)]$
$\wedge [(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \wedge \neg R)]$

{Distributive Law}

$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R)$
$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \wedge \neg R)$

$\{ P \vee R \vee \neg Q \Leftrightarrow \neg Q \vee P \vee R$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\{ P \vee R \vee Q \Leftrightarrow P \vee Q \vee R$

$S \Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

is required PCNF.

To find PDNF

$\neg S \Leftrightarrow$ The Remaining maxterms of P, Q & R.

The maxterms of P, Q, R are

$(P \vee Q \vee R)$, $(\neg P \vee Q \vee R)$, $(P \vee \neg Q \vee R)$, $(P \vee Q \vee \neg R)$

$(\neg P \vee \neg Q \vee R)$, $(\neg P \vee Q \vee \neg R)$, $(P \vee \neg Q \vee \neg R)$, $(\neg P \vee \neg Q \vee \neg R)$

$\therefore \neg S \Leftrightarrow (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\neg(\neg S) \Leftrightarrow \neg[(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)]$

$S \Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

is required PDNF.

Method-2

Let $S \Leftrightarrow (\neg P \to R) \wedge (Q \leftrightarrow P)$

Here $A = \neg P \to R$; $B = Q \leftrightarrow P$, $S = A \wedge B$

P	Q	R	$\neg P$	$A = \neg P \to R$	$B = Q \leftrightarrow P$	$S = A \wedge B$	minterms	maxterms
T	T	T	F	T	T	T	$P \wedge Q \wedge R$	
T	T	F	F	T	T	T	$P \wedge Q \wedge \neg R$	
T	F	T	F	T	F	F		$\neg P \vee Q \vee \neg R$
T	F	F	F	T	F	F		$\neg P \vee Q \vee R$
F	T	T	T	T	F	F		$P \vee \neg Q \vee \neg R$
F	T	F	T	F	F	F		$P \vee \neg Q \vee R$
F	F	T	T	T	T	T	$\neg P \wedge \neg Q \wedge R$	
F	F	F	T	F	T	F		$P \vee Q \vee R$

The PDNF $= (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$

The PCNF $= (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$
$\wedge (P \vee Q \vee R)$

(5) obtain PCNF and PDNF of the formula
$(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

Soln) Given that $S: (\neg P \vee \neg Q) \longrightarrow (P \leftrightarrow \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \leftrightarrow \neg Q$	S	minterms	maxterms
T	T	F	F	F	F	T	$P \wedge Q$	
T	F	F	T	T	T	T	$P \wedge \neg Q$	
F	T	T	F	T	T	T	$\neg P \wedge Q$	
F	F	T	T	T	T	F		$P \vee Q$

The PCNF : $P \vee Q$

The PDNF : $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

(6) obtain PDNF of $P \rightarrow [P \wedge (Q \rightarrow P)]$

Soln) Given that $P \rightarrow [P \wedge (Q \rightarrow P)]$

$\Leftrightarrow \neg P \vee [P \wedge (Q \rightarrow P)]$ ~~Demorgans law~~

$\Leftrightarrow \neg P \vee [P \wedge (\neg Q \vee P)]$

$\Leftrightarrow \neg P \vee (P \wedge \neg Q) \vee (P \wedge P)$

$\Leftrightarrow (\neg P \wedge T) \vee (P \wedge \neg Q) \vee (P \wedge P)$

$\Leftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge \neg Q) \vee [P \wedge (Q \vee \neg Q)]$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee (P \wedge Q) \vee (P \wedge \neg Q)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee (P \wedge Q)$

This is required PDNF.

(7) obtain the PDNF and PCFN of

$P \vee [\neg P \to [Q \vee (\neg Q \to R)]]$.

Solu:

Given that $S: P \vee [\neg P \to [Q \vee (\neg Q \to R)]]$

$$\Leftrightarrow P \vee [P \vee [Q \vee (\neg Q \to R)]]$$

$$\Leftrightarrow P \vee [P \vee [Q \vee (Q \vee R)]]$$

$$\Leftrightarrow (P \vee P) \vee (Q \vee Q) \vee R$$

$$S \Leftrightarrow (P \vee Q \vee R)$$

This is required PCNF.

To find PDNF

$\neg S$ = The Remaining maxterms of P, Q & R

The Possible maxterms of P, Q, R are

$$\neg S = (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

$$\neg\neg S = \neg[(\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee \neg Q)]$$

$$S = (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R).$$

Chapter -1.4
Rules of Inference

Rule P: A Premise may be introduced at any Point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is a tautologically implied by any one or more of the Preceding formulas in the derivation.

Rule CP: If we can derive S from R and a set of Premises, then we can derive $R \to S$ from the set of Premises alone

Eg-① Demonstrate that R is a valid inference from the Premises $P \to Q$, $Q \to R$ and P.

Soln Given that $P \to Q$, $Q \to R$ and $P \Rightarrow R$

steps	statements	Rules & formula
(1)	P	Rule P
(2)	$P \to Q$	Rule P
(3)	Q	Rule T{1,2} $P, P \to Q \Rightarrow Q$
(4)	$Q \to R$	Rule P
(5)	R	Rule T{3,4} $P, P \to Q \Rightarrow Q$

② Show that $J \wedge S$ logically follows from the Premises $P \to Q$, $Q \to \neg R$, R, $P \vee (J \wedge S)$

Sol:

Given that $P \to Q,\ Q \to \neg R,\ R,\ P \vee (J \wedge S) \Rightarrow J \wedge S$

Steps	Statements	Rules & formula
(1)	R	Rule P
(2)	$Q \to \neg R$	Rule P $\quad \neg(\neg R) = R$
(3)	$\neg Q$	Rule T{1,2} $\quad P \to Q, \neg Q \Rightarrow \neg P$
(4)	$P \to Q$	Rule P
(5)	$\neg P$	Rule T{3,4} $\quad P \to Q, \neg Q \Rightarrow \neg P$
(6)	$P \vee (J \wedge S)$	Rule P
(7)	$J \wedge S$	Rule T{5,6} $\quad P \vee Q, \neg P \Rightarrow Q$

③ Show that $R \to S$ can be derived from the Premises $P \to (Q \to S)$, $\neg R \vee P$ and Q

Soln: Given that $P \to (Q \to S),\ \neg R \vee P,\ Q \Rightarrow R \to S$

Steps	Statements	Rules & formula
(1)	R	Rule P {Assumed Premises}
(2)	$\neg R \vee P$	Rule P
(3)	$R \to P$	Rule T{2} $\quad P \to Q \Rightarrow \neg P \vee Q$
(4)	P	Rule T{1,3} $\quad P, P \to Q \Rightarrow Q$
(5)	$P \to (Q \to S)$	Rule P
(6)	$Q \to S$	Rule T{4,5} $\quad P, P \to Q \Rightarrow Q$
(7)	Q	Rule P
(8)	S	Rule T{6,7} $\quad P, P \to Q \Rightarrow Q$
(9)	$R \to S$	Rule CP

(4) Show that $\neg P$ is valid from $\neg(P \wedge \neg Q) \wedge (\neg Q \vee R) \wedge \neg R$

Soln)

Given that $\neg(P \wedge \neg Q), (\neg Q \vee R), \neg R \Rightarrow \neg P$

Steps	Statements	Rules & formula
(1)	$\neg R$	Rule P
(2)	$\neg Q \vee R$	Rule P
(3)	$Q \rightarrow R$	Rule T {2} $P \rightarrow Q \Rightarrow \neg P \vee R$
(4)	$\neg Q$	Rule T {1,3} $P \rightarrow Q, \neg Q \Rightarrow \neg P$
(5)	$\neg(P \wedge \neg Q)$	Rule P
(6)	$\neg P \vee Q$	Rule T {5} $\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$
(7)	$P \rightarrow Q$	Rule T {6} $P \rightarrow Q \Rightarrow \neg P \vee Q$
(8)	$\neg P$	Rule T {4,7} $P \rightarrow Q, \neg Q \Rightarrow \neg P$

(5) Using indirect method, show that $P \rightarrow Q, Q \rightarrow R, P \vee R \Rightarrow R$

Soln)

Given that $P \rightarrow Q;\ Q \rightarrow R;\ P \vee R \Rightarrow R$.

Steps	Statements	Rules & formula
(1)	$\neg R$	Rule P & Assumed Premises
(2)	$Q \rightarrow R$	Rule P
(3)	$\neg Q$	Rule T {1,2} $P \rightarrow Q, \neg Q \Rightarrow \neg P$
(4)	$P \rightarrow Q$	Rule P
(5)	$\neg P$	Rule T {3,4} $P \rightarrow Q, \neg Q \Rightarrow \neg P$
(6)	$P \vee R$	Rule P
(7)	$\neg P \rightarrow R$	Rule T {6} $\neg P \rightarrow Q \Rightarrow P \vee Q$
(8)	R	Rule T {5,7} $P \rightarrow Q, P \Rightarrow Q$
(9)	$R \wedge \neg R$	Rule T {1,8} which is contradiction

⑥ Using indirect method, show that

$R \to \neg Q,\ R \vee S,\ S \to \neg Q,\ P \to Q \Rightarrow \neg P$

Soln:

Given that $R \to \neg Q,\ R \vee S,\ S \to \neg Q,\ P \to Q \Rightarrow \neg P$

steps	statements	Rules & formula
(1)	P	Rule P {Assumed Premises
(2)	$P \to Q$	Rule P
(3)	Q	Rule T {1,2} $P, P \to Q \Rightarrow Q$
(4)	$R \to \neg Q$	Rule P
(5)	$\neg R$	Rule T {3,4), $P \to Q, \neg Q \Rightarrow \neg P$
(6)	$R \vee S$	Rule P
(7)	$\neg R \to S$	Rule T {6} $\neg P \to Q \Rightarrow P \vee Q$
(8)	S	~~Rule T {5}~~ Rule T {5,7} $P, P \to Q \Rightarrow Q$
(9)	$S \to \neg Q$	Rule P
(10)	$\neg Q$	Rule T {8,9} $P, P \to Q \Rightarrow Q$
(11)	$Q \wedge \neg Q$	Rule T {3,10} $P, Q \Rightarrow P \wedge Q$

which is Contradiction

$\neg P$ is derived from $R \to \neg Q,\ R \vee S,\ S \to \neg Q,\ P \to Q$

⑦ Show that the following set of premises are inconsistent. $P \to Q$, $P \to R$, $Q \to \neg R$, P

Sol/w

Given that $P \to Q$, $P \to R$, $Q \to \neg R$, P

Steps	statements	Rules & formula
(1)	P	Rule P
(2)	$P \to Q$	Rule P
(3)	Q	Rule T {1,2} $P, P \to Q \Rightarrow Q$
(4)	$Q \to \neg R$	Rule P
(5)	$\neg R$	Rule T {3,4} $P, P \to Q \Rightarrow Q$
(6)	$P \to R$	Rule P
(7)	$\neg P$	Rule T {5,6} $P \to Q, \neg Q \Rightarrow \neg P$
(8)	$\neg P \wedge P = F$	Rule T {1,7} $P, Q \Rightarrow P \wedge Q$

∴ Given premises are inconsistent.

⑧ Show that the following set of premises are inconsistent $P \to Q$, $Q \to R$, $R \to S$, $S \to \neg R$ and $P \wedge S$.

Sol/w

Given that $P \to Q$, $Q \to R$, $R \to S$, $S \to \neg R$ & $P \wedge S$.

steps	statements	Rules & formula
(1)	$P \to Q$	Rule P
(2)	$Q \to R$	Rule P
(3)	$P \to R$	Rule T {1,2} chain rule $P \to Q, Q \to R \Rightarrow P \to R$
(4)	$R \to S$	Rule P
(5)	$P \to S$	Rule T {3,4} chain rule $P \to R, R \to S \Rightarrow P \to S$

(6)	$S \rightarrow \neg R$	Rule P
(7)	$P \rightarrow \neg R$	Rule T {5,6} chain rule $P \rightarrow S,\ S \rightarrow \neg R \Rightarrow P \rightarrow \neg R$
(8)	$P \wedge S$	Rule P
(9)	P	Rule T {8} $P \wedge Q \Rightarrow P, Q$
(10)	$\neg R$	Rule T {7,9} $P,\ P \rightarrow Q \Rightarrow Q$
(11)	R	Rule T {3,9} $P,\ P \rightarrow Q \Rightarrow Q$
(12)	$\neg R \wedge R = F$	Rule T {10,11} $P \wedge Q \Rightarrow P, Q$.

Given Permises are inconsistent

(9) Show that the following statements constitute a valid argument. If there was rain, then traveling was difficult. If they had umbrella, then traveling was not difficult. They had umbrella. Therefore there was no rain

Soln.

Let P = There was rain, Q = Traveling was difficult

R = They had umbrella

then the Permises are $P \rightarrow Q,\ R \rightarrow \neg Q,\ R,\ \neg P$

Conclusion : $\neg P$

Steps	statements	Rules & formula
(1)	R	Rule P
(2)	$R \rightarrow \neg Q$	Rule P
(3)	$\neg Q$	Rule T {1,2} $P \rightarrow \neg Q,\ P \Rightarrow \neg Q$
(4)	$P \rightarrow Q$	Rule P
(5)	$\neg P$	Rule T {3,4} $P \rightarrow Q,\ \neg Q \Rightarrow \neg P$

(10) Show that the following set of Premises are inconsistent

1) If Jack misses many classes through illness, then he fails high school.
2) If Jack fails high school, then he is uneducated
3) If Jack reads a lot of book, then he is not uneducated
4) Jack misses many classes through illness and reads a lot of book.

Sol:

Let P: Jack misses many classes

Q: Jack fails high school

R: Jack is uneducated

S: Jack reads lot of books.

then, the Permises are, $P \to Q$, $Q \to R$, $S \to \neg R$, $P \wedge S$

steps	statements	Rules & formula
(1)	$P \to Q$	Rule P
(2)	$Q \to R$	Rule P
(3)	$P \to R$	Rule T {1,2} by chain rule
(4)	$S \to \neg R$	Rule P
(5)	$R \to \neg S$	Rule T {4} $P \to Q \Leftrightarrow \neg Q \to P$
(6)	$P \to \neg S$	Rule T {3,5} by chain rule
(7)	$P \wedge S$	Rule P
(8)	$\neg P \vee \neg S$	Rule T {6} $P \wedge S \Rightarrow \neg P \vee Q$
(9)	$\neg(P \wedge S)$	Rule T {8} Demorgan's law
(10)	$(P \wedge S) \wedge \neg(P \wedge S)$	Rule T {7,9} $P, \neg P \Rightarrow P \wedge \neg Q$

F ∴ The given Permises are inconsistent

Chapter-1.5
Predicate and Quantifiers

Predicate:

(i) $\underbrace{\text{Suresh}}_{x}$ $\underbrace{\text{is a boy}}_{B}$: $B(x)$
(ii) $\underbrace{\text{Priya}}_{x}$ $\underbrace{\text{is rich}}_{R}$: $R(x)$

} 1-Place Predicate

(iii) $\underbrace{\text{Prakash}}_{x}$ $\underbrace{\text{is taller than}}_{T}$ $\underbrace{\text{Suresh}}_{y}$: $T(x,y)$ – 2-Place Predicate

Quantifiers: [used to quantity the nature of variable]

① universal quantifier:
② Existential quantifier

⇒ universal quantifier [$\forall(x)$]

for all x, for every x, for Each x,
Everything x is such that.

Eg:- ① All milk are white

$\forall x$, $\underbrace{x \text{ is milk}}_{M(x)}$ then $\underbrace{x \text{ is white}}_{W(x)}$

$(\forall x)(M(x) \rightarrow W(x))$

Eg-② Everthing $\underbrace{\text{is green}}_{G(x)}$

$\forall x$, x is green

$(\forall x)\ G(x)$.

$\Rightarrow$ Existential Quantifier : $(\exists x)$

for some x, some x such that,

there exists an x, there is atleast one such that.

Eg: Some students are tall

$\underbrace{x \text{ is student}}_{S(x)}$, $\underbrace{x \text{ is tall}}_{T(x)}$

$(\exists x)(S(x) \wedge T(x))$

Rules of Generalization and Specification

① Universal Specification [Rule US]

$(\forall x)\ P(x) \Rightarrow P(y)$

② Existential Specification [Rule ES]

$(\exists x)\ P(x) \Rightarrow P(y)$

③ Universal Generalization [Rule UG]

$P(y) \Rightarrow (\forall x)\ P(x)$

④ Existential Generalization [Rule EG]

$P(y) \Rightarrow (\exists x)\ P(x)$

⑤ $\neg[(\forall x)\ P(x)] \Rightarrow (\exists x) \neg P(x)$

⑥ $\neg[(\exists x)\ P(x)] \Rightarrow (\forall x) \neg P(x)$

Eg-① Symbolise : For every x, there exists a y such that $x^2+y^2 \geq 100$

Soln

$$(\forall x)(\exists y)(x^2+y^2 \geq 100)$$

Eg-② Give the symbolic form of the statement "Every book with a blue cover is a maths book".

Soln:

Let $S(x)$: x is every book with a blue cover

$P(x)$: maths book

$$(\forall x)(S(x) \rightarrow P(x))$$

Eg-③ Write the following sentence in symbolic form.

"All lions are fierce"

"Some lions do not drink coffee"

"Some fierce creatures do not drink coffee"

Soln:

Let $P(x)$: x is a lion

$Q(x)$: x is fierce

$R(x)$: x is drinks coffee

We get

(i) $(\forall x)(P(x) \rightarrow Q(x))$

(ii) $(\exists x)[P(x) \wedge \neg R(x)]$

(iii) $(\exists x)[Q(x) \wedge \neg R(x)]$

Eg-④ Write the following sentences in symbolic form.

"All humming birds are richly colored"

"No large birds live on honey"

"Birds that do not live on honey are dull in color"

"Humming birds are small."

Solⁿ:

Let $P(x)$: x is a humming bird

$Q(x)$: x is a large

$R(x)$: x is lives on honey

$S(x)$: x is richly colored.

~~we get~~ we get. (i) $\forall x\,[P(x) \rightarrow S(x)]$

(ii) $\neg \exists x\,[Q(x) \wedge R(x)]$

(iii) $\forall x\,[\neg R(x) \rightarrow \neg S(x)]$

iv) $\forall x\,[P(x) \rightarrow \neg Q(x)]$

Eg-⑤ Write each of the following in symbolic form

(a) All men are good

(b) No men are good

(c) Some men are good

(d) Some men are not good

Soln: Let $M(x)$: x is a man

$G(x)$: x is Good.

(a) means "for all x, if x is a man, then x is good"

$$\forall x \, [M(x) \longrightarrow G(x)]$$

(b) means "for all x, if x is a man, then x is not good"

$$\forall x \, [M(x) \longrightarrow \neg G(x)]$$

(c) means, "there is an x, such that x is a man and x is good"

$$\exists x \, [M(x) \wedge G(x)]$$

(d) means, "there is an x, such that x is man and x is not good"

$$\exists x \, [M(x) \wedge \neg G(x)].$$

① Prove that $(\exists x)[P(x) \wedge Q(x)] \Rightarrow (\exists x)\, P(x) \wedge (\exists x)\, Q(x)$

Soln: Given that $\exists(x)\,[P(x) \wedge Q(x)] \Longrightarrow \exists(x)\, P(x) \wedge \exists(x)\, Q(x)$

step	statements	Rules
(1)	$\exists(x)[P(x) \wedge Q(x)]$	Rule P
(2)	$P(y) \wedge Q(y)$	Rule ES {1}
(3)	$P(y)$	Rule T {2} , $P \wedge Q \Rightarrow P, Q$
(4)	$Q(y)$	Rule T {2}
(5)	$\exists(x)\, P(x)$	Rule EG {3}
(6)	$\exists(x)\, Q(x)$	Rule EG {4}
(7)	$\exists(x) P(x) \wedge \exists(x)\, Q(x)$	Rule T {5,6} $P, Q \Rightarrow P \wedge Q$

(2) Is the following conclusion validly derivable from the premises given?

If $\forall(x)[P(x) \rightarrow Q(x)]$; $\exists(y)\,P(y)$; then $\exists(z)\,Q(z)$

Soln

Given that $\forall(x)[P(x) \rightarrow Q(x)]$; $\exists(y)\,P(y)$; then $\exists(z)\,Q(z)$.

Steps	Statements	Rules
(1)	$\exists(y)\,P(y)$	Rule P
(2)	$P(a)$	Rule ES {1}
(3)	$\forall(x)[P(x) \rightarrow Q(x)]$	Rule P
(4)	$P(a) \rightarrow Q(a)$	Rule US {3}
(5)	$Q(a)$	Rule T {2, 4}, $P, P \rightarrow Q \Rightarrow Q$
(6)	$\exists(z)\,Q(z)$	Rule EG {5}

(3) Show that $\exists(x)\,P(x) \rightarrow \forall(x)\,Q(x) \Rightarrow \forall(x)[P(x) \rightarrow Q(x)]$

Soln Given that $\exists(x)\,P(x) \rightarrow \forall(x)\,Q(x) \Rightarrow \forall(x)[P(x) \rightarrow Q(x)]$

Steps	Statement	Rules
(1)	$P(y)$	Rule P (Assumed)
(2)	$\exists(x)\,P(x) \rightarrow \forall(x)\,Q(x)$	Rule P
(3)	$\exists(x)\,P(x)$	Rule EG {1}
(4)	$\forall(x)\,Q(x)$	Rule T {2, 3} $P, P \rightarrow Q \Rightarrow Q$
(5)	$Q(y)$	Rule US {4}
(6)	$P(y) \rightarrow Q(y)$	Rule CP {1, 5}
(7)	$\forall(x)[P(x) \rightarrow Q(x)]$	Rule UG {6}

④ Using CP or otherwise obtain the following implication

$\forall(x)[P(x) \to Q(x)], \forall(x)[R(x) \to \neg Q(x)] \Rightarrow \forall(x)[R(x) \to \neg P(x)]$

Soln)

Given that $\forall(x)[P(x) \to Q(x)], \forall(x)[R(x) \to \neg Q(x)] \Rightarrow \forall(x)[R(x) \to \neg P(x)]$

step	statements	Rules
(1)	$R(y)$	Rule P (Assumed)
(2)	$\forall(x)[R(x) \to \neg Q(x)]$	Rule P
(3)	$R(y) \to \neg Q(y)$	Rule US{2}
(4)	$\neg Q(y)$	Rule T{1,3} $P, P \to Q \Rightarrow Q$
(5)	$\forall(x)[P(x) \to Q(x)]$	Rule P
(6)	$P(y) \to Q(y)$	Rule US{5}
(7)	$\neg P(y)$	Rule T {4,6} $P \to Q, \neg Q \Rightarrow \neg P$
(8)	$R(y) \to \neg P(y)$	Rule CP{1,7}
(9)	$\forall(x)[R(x) \to \neg P(x)]$	Rule UG{8}

⑤ Show that the Premises "A student in this class has not read the book", and Everyone in this class Passed the first Exam" imply the conclusion "Someone who Passed the first Exam has not read the book.

Soln:

Let $P(x)$: x is in this class

$Q(x)$: x has read the book

$R(x)$: x Passed the first Exam

To find

$\exists(x)[P(x) \wedge \neg Q(x)]$; $\forall(x)[P(x) \to R(x)] \Rightarrow \exists(x)[R(x) \wedge \neg Q(x)]$

steps	statements	Rules
(1)	$\exists(x)[P(x) \wedge \neg Q(x)]$	Rule P
(2)	$P(y) \wedge \neg Q(y)$	Rule ES {1}
(3)	$P(y)$	Rule T {2}, $P \wedge Q \Rightarrow P, Q$
(4)	$\neg Q(y)$	Rule T {2}, $P \wedge Q \Rightarrow P, Q$
(5)	$\forall(x)[P(x) \to R(x)]$	Rule P
(6)	$P(y) \to R(y)$	Rule US {5}
(7)	$R(y)$	Rule T {3,6} $P, P \to Q \Rightarrow Q$
(8)	$R(y) \wedge \neg Q(y)$	Rule T {4,7} $P, Q \Rightarrow P \wedge Q$
(9)	$\exists(x)[R(x) \wedge \neg Q(x)]$	Rule EG {8}

6. Show that the Premises "one student in this class knows how to write Programs in JAVA" and "Everyone who knows how to write Programs in JAVA can get a high-Paying Job" imply the conclusion "some one in this class can get a high-Paying Job".

Soln:

Let $P(x)$: x is in this class

$Q(x)$: x knows "JAVA" Programming

$R(x)$: x can get a high-Paying Job.

To find :-

$\exists(x)[P(x) \wedge Q(x)],\ \forall(x)[Q(x) \rightarrow R(x)] \Rightarrow \exists(x)[P(x) \wedge R(x)]$

steps	statements	Rules
(1)	$\exists(x)[P(x) \wedge Q(x)]$	Rule P
(2)	$P(y) \wedge Q(y)$	Rule ES {1}
(3)	$P(y)$	Rule T {2} $P \wedge Q \Rightarrow P, Q$
(4)	$Q(y)$	Rule T {2}
(5)	$\forall(x)[Q(x) \rightarrow R(x)]$	Rule P
(6)	$Q(y) \rightarrow R(y)$	Rule US {5}
(7)	$R(y)$	Rule T{4, 6} $P, P \rightarrow Q \Rightarrow Q$
(8)	$P(y) \wedge R(y)$	Rule T {3,7} $P, Q \Rightarrow P \wedge Q$
(9)	$\exists(x)[P(x) \wedge R(x)]$	Rule EG {8}

(7) Establish the validity of the following argument:
" All integer are rational number. Some integer are Powers of 2. Therefore, Some rational number are Powers of 2.

Soln:

let $P(x)$: x is an integer

$Q(x)$: x is an rational number

$R(x)$: x is a Power of 2.

To find:

$\forall(x)\,[P(x) \rightarrow Q(x)],\ \exists(x)\,[P(x) \wedge R(x)] \Rightarrow \exists(x)\,[Q(x) \wedge R(x)]$

steps	statements	Rules
(1)	$\forall(x)\,[P(x) \rightarrow Q(x)]$	Rule P
(2)	$P(y) \rightarrow Q(y)$	Rule US {1}
(3)	$\exists(x)\,[P(x) \wedge R(x)]$	Rule P
(4)	$P(y) \wedge R(y)$	Rule ES {3}
(5)	$P(y)$	Rule T {4}, $P \wedge Q \Rightarrow P, Q$
(6)	$R(y)$	Rule T {4} $P \wedge Q \Rightarrow P, Q$
(7)	$Q(y)$	Rule T {2,5}, $P, P \rightarrow Q \Rightarrow Q$
(8)	$Q(y) \wedge R(y)$	Rule T {6,7} $P, Q \Rightarrow P \wedge Q$
(9)	$\exists(x)\,[Q(x) \wedge R(x)$	Rule EG {8}

Chapter - 1.6

Introduction to Proofs

TYPE - 1 DIRECT PROOFS

(i) Hypothesis: First we assume that P is true

(ii) Analysis: starting with the hypothesis and employing the rules / laws of logic and other known facts, infer that Q is true.

(iii) Conclusion: $P \rightarrow Q$ is true.

① Give a direct proof of the statement. "The square of an odd integer is an odd integer".

Soln: Given that: "The square of an odd integer is an odd integer"

ie "If n is an odd integer, then n^2 is an odd integer"

Let P: n is an odd integer

Q: n^2 is an odd integer

Hypothesis: Let us assume that P is true

ie, n is an odd integer is true

Analysis: $n = 2k+1$ (by definition of odd integer)

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 1 + 4k$$
$$= 4k^2 + 4k + 1$$
$$= \cancel{4}2(2k^2 + 2k) + 1$$
$$= 2(2k^2 + 2k) + 1$$

Conclusion: We observe that R.H.S value is not divisible by 2.

$\therefore$ n^2 is not divisible by 2.

n^2 is an odd integer

ie, $P \to Q$ is true.

(2) Give a direct proof of "The sum of two odd integer is even".

Soln: Given that: "The sum of two odd integer is Even".

ie, If n is odd and m is odd then $n+m$ is Even integer

P: n is an odd integer and m is an odd integer

Q: $n+m$ is an odd integer.

Hypothesis: Assume that P is true

Analysis: If n is an odd integer

then $n = 2k+1$ for some integer k.

If m is an odd integer

then $m = 2l+1$, for some integer l

$$n+m = (2k+1) + (2l+1)$$
$$= 2k + 2l + 2$$
$$= 2(k+l+1)$$

Conclusion: We observe that the R.H.S value is divisible by 2.

This means that $n+m$ is an even integer

i.e., $P \to Q$ is true.

(5) use a direct proof to show that "Every odd integer is the difference of two squares."

Soln:

If n is an odd integer then $n = s^2 - t^2$

Let P: n is an odd integer

Q: $n = s^2 - t^2$

Hypothesis: Assume that P is true.

Analysis: If n is an integer then,

$n = 2k+1$, where k is some integer

$$n = k^2 + (2k+1) - k^2$$
$$= (k^2 + 2k + 1) - k^2$$
$$= (k+1)^2 - k^2$$
$$= s^2 - t^2, \text{ where } s = k+1, t = k.$$

Conclusion:

We observe that the R.H.S value is the difference of two squares

$P \to Q$ is true.

TYPE-2 Proof by Contraposition

(i) Hypothesis: $p \to q$ its contrapositive $\neg q \to \neg p$ are logically equivalent. So assume that $\neg q$ is true

(ii) Analysis: starting with the hypothesis and employing the rules / laws of logic and other known facts, infer that $\neg p$ is true.

(iii) Conclusion: $\neg q \to \neg p$ is true.

① Prove that if n is an integer and $5n+2$ is odd, then n is odd.

Soln Let p: n is an integer and $5n+2$ is odd

q: n is odd.

To prove $p \to q$ is true it is enough to prove $\neg q \to \neg p$ is true

Hypothesis: Since $p \to q$ its contrapositive $\neg q \to \neg p$ are logically equivalent

So assume that $\neg q$ is true

ie n is even

Analysis: If n is even, then $n = 2k$, for some integer k.

$$\therefore 5n+2 = 5(2k)+2$$
$$= 10k+2$$
$$= 2[5k+1]$$

Conclusion: we observe that R.H.S value of $5n+2$ is divisible by 2.

This means that $5n+2$ is an even integer

ie, $\neg P$ is true

$\therefore \neg Q \rightarrow \neg P$ is true.

② Show that if n is an integer and n^3+5 is odd, then n is even using a proof by contraposition.

Soln:

Let P: n is an integer and n^3+5 is odd

Q: n is even

To prove $P \rightarrow Q$ is true, it is enough to prove that $\neg Q \rightarrow \neg P$ is true.

Hypothesis: Since $P \rightarrow Q$ contrapositive $\neg Q \rightarrow \neg P$ are logically equivalent

so assume that $\neg Q$ is true

ie, n is odd.

Analysis: If n is odd then $n = 2k+1$ for some integer k

$$\therefore n^3+5 = (2k+1)^3+5$$
$$= (2k)^3 + 3(2k)^2(1) + 3(2k)(1)^2 + (1)^3 + 5$$
$$= 8k^3 + 3(4k^2) + 6k + 1 + 5$$
$$= 8k^3 + 12k^2 + 6k + 6$$
$$= 2(4k^3 + 6k^2 + 3k + 3)$$

Conclusion: We observe that R.H.S value of n^3+5 is divisible by 2.

This means that n^3+5 is an Even integer

i.e, $\neg P$ is true

$\neg Q \rightarrow \neg P$ is true

TYPE-3 Problem based on Contradiction.

① Proof by contradiction of the following statement "For Every integer n, if n^2 is odd, then n is odd."

Soln:

Let P: For Every integer n, if n^2 is odd

Q: n is odd.

Hypothesis: Assume that $P \rightarrow Q$ is false

i.e, Assume that P is true and Q is false

i.e, n is not odd $\Rightarrow$ n is Even

Analysis: If n is even then $n = 2k$ for some integer k.

$$n^2 = (2k)^2 = 4k^2$$

Conclusion:

We conclude that R.H.S of n^2 is divisible by 2

This means that n^2 is Even

This contradicts the assumption P is true

The given Conditional $P \rightarrow Q$ is true.

(2) Give a proof contradiction of the theorem "If $3n+2$ is odd, then n is odd."

Soln: Let P: $3n+2$ is odd

Q: n is odd

Hypotheses: Assume that $P \to Q$ is false

ie, Assume that P is true and Q is false

ie, n is not odd $\Rightarrow$ n is Even

Analysis:

If n is Even then $n = 2k$ for some integer k.

$$3n+2 = 3(2k)+2$$
$$= 6k+2$$
$$= 2(3k+1)$$

Conclusion:

We observe that the R.H.S value of $3n+2$ is divisible by 2.

This means that $3n+2$ is Even

This contradicts the assumption P is true

The given conditional $P \to Q$ is true.

(5) Prove that $\sqrt{2}$ is irrational by giving proof using contradiction.

Soln:

~~Suppose~~ Given that $\sqrt{2}$ is irrational

Let us assume $\sqrt{2}$ is rational

$\therefore \sqrt{2} = p/q$, for $p, q \in \mathbb{Z}$, $q \neq 0$, p & q have no common divisor

$\Rightarrow \sqrt{2}q = p$

$2q^2 = p^2$

ie, $p^2 = 2q^2 \longrightarrow (1)$

$\therefore p^2$ is Even $\Rightarrow p$ is Even

Let $p = 2(m)$, m is an integer

$(1) \Rightarrow p^2 = 2q^2$

$(2m)^2 = 2q^2$

$4m^2 = 2q^2$

$2q^2 = 4m^2$

$q^2 = \frac{\cancel{4}^{2}m^2}{\cancel{2}}$

$q^2 = 2m^2$

$\therefore q^2$ is Even $\Rightarrow q$ is Even

Let $q = 2(n)$, n is an integer

$\therefore \sqrt{2} = p/q = \frac{2m}{2n}$

Here p, q have a common divisor.

which is $\Rightarrow\Leftarrow$

$\therefore \sqrt{2}$ is irrational number

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SANCET

2024–2025

MA3354-
DISCRETE MATHEMATICS

UNIT-2

COMBINATORICS

Prepare by

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UNIT-2

COMBINATORICS

Mathematical induction – Strong induction and well ordering – The basics of counting – The pigeonhole principle – Permutations and combinations – Recurrence relations – Solving linear recurrence relations – Generating functions – Inclusion and exclusion principle and its applications.

UNIT-2

COMBINATORICS

Chapter-2.1 [Mathematical Induction]

Principle of Mathematical Induction

Let $P(n)$ be a statement or Proposition involving for all Positive integers n. Then we complete two steps.

Step-1 :- If $P(1)$ is true

Step-2 :- If $P(k+1)$ is true on the assumption that $P(k)$ is true

Problem-1 Prove by induction $1+2+3+ \cdots +n = \frac{n(n+1)}{2}, n \geq 1$.

Soln:

Let $P(n) : 1+2+3+ \cdots +n = \frac{n(n+1)}{2}, n \geq 1$

Step-1:- To Prove $P(1)$ is true

for, $\boxed{n=1}$, we have

$$1 = \frac{1(1+1)}{2} \Rightarrow 1 = \frac{\cancel{2}}{\cancel{2}} \Rightarrow 1=1$$

So, $P(1)$ is true.

Step-2:- Assume that $P(k)$ is true for any +ve integer 'k'

ie, $1+2+3+4+ \cdots +k = \frac{k(k+1)}{2}$.

Step-3 To Prove $P(k+1)$ is true

ie $P(k+1) = \frac{(k+1)(k+2)}{2}$

$$[1+2+3+\cdots+k]+k+1 = \left[\frac{k(k+1)}{2}\right]+k+1$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

which is $P(k+1)$

That is $P(k+1)$ is true whether $P(k)$ is true

By the Principle of mathematical induction $P(n)$ is true for all positive integer n.

Problem-2 Show that $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \geq 1$ by mathematical induction.

Soln: Let $P_n: 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \geq 1$

Step-1: To Prove $P(1)$ is true

for $\boxed{n=1}$

$$1 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 6/6 = 1$$

So, $P(1)$ is true

Step-2: Assume that $P(k)$ is true

i.e. $1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Step-3: To Prove $P(k+1)$ is true

$2k+1$
$= 2(k+1)+1$
$= 2k+2+1$
$= 2k+3$

i.e. $P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$

$$[1^2+2^2+3^2+\cdots+k^2]+(k+1)^2 = \left[\frac{k(k+1)(2k+1)}{6}\right]+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$2k^2+7k+6$
$2k^2+7k+6$
$= 2k^2+4k+3k+6$
$= 2k(k+2)+3(k+2)$
$= (2k+3)(k+2)$

which is $P(k+1)$

That is $P(k+1)$ is true whenever $P(k)$ is true

By mathematical induction $P(n)$ is true for all positive integer n.

Problems-(3)

Prove the formula for the sum of first n cubes using the mathematical induction.

$$S_n = \left[\frac{n(n+1)}{2}\right]^2$$

(OR)

Show that $1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$, $n \geq 1$.

Soln:

Let $P(n) : 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Step-1 To Prove P(1) is true

$$1^3 = \left[\frac{1(1+1)}{2}\right]^2 \Rightarrow 1 = \left[\frac{1(2)}{2}\right]^2 = \left(\cancel{\frac{2}{2}}\right)^2 = (1)^2 = 1$$

So, P(1) is true

Step-2 Assume that P(k) is true

i.e. $1^3+2^3+3^3+\cdots+k^3 = \left[\frac{k(k+1)}{2}\right]^2$

Step-3: To Prove P(k+1) is true

i.e. $P(k+1) = \left[\frac{(k+1)(k+2)}{2}\right]^2$

$$\left[1^3+2^3+3^3+\cdots+k^3\right]+(k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3$$

$$= \frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$= \frac{K^2(K+1)^2 + 4(K+1)^3}{4}$$

$$= \frac{(K+1)^2[K^2 + 4(K+1)]}{4}$$

$$= \frac{(K+1)^2[K^2 + 4K + 4]}{4}$$

$$= \frac{(K+1)^2[(K+2)(K+2)]}{4}$$

$$= \frac{(K+1)^2(K+2)^2}{4}$$

$$= \left[\frac{(K+1)(K+2)}{2}\right]^2$$

$K^2 + 4K + 4 = 0$

$$\begin{array}{c} 4 \\ \hline 2 \mid 2 \\ \hline 4 \end{array}$$

which is $P(K+1)$

That is $P(K+1)$ is true whenever $P(K)$ is true.

(4) Show that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Soln:- Let $P(n)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Step-1:- To prove $P(1)$ is true

$$\frac{1}{1\cdot 2} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$

So, $P(1)$ is true

<u>Step-2:-</u> Assume that $P(k)$ is true

i.e. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

<u>Step-3:-</u> To Prove $P(k+1)$ is true

i.e, $P(k+1)=\frac{k+1}{(k+1)+1}=\frac{k+1}{k+2}$

$$\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$\left\{ \begin{array}{l} k^2+2k+1=0 \\ =(k+1)(k+1) \quad \begin{array}{c} 1 \\ \hline 1 \mid 1 \\ \hline 2 \end{array} \end{array} \right.$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$$=\frac{k+1}{k+2}$$

Which is $P(k+1)$

That is $P(k+1)$ is true whenever $P(k)$ is true.

Problem-(5) Use mathematical induction to show that $2^n < n!$ for every positive integer n with $n \geq 4$.

Soln:

Let $P(n): 2^n < n!, \; n \geq 4$

Step-1: To prove $P(4)$ is true

For $\boxed{n=4}$ $2^4 < 4!$

$16 < 4 \times 3 \times 2 \times 1$

$16 < 24$

So, $P(4)$ is true

Step-2: Assume that $P(K)$ is true

i.e. $2^K < K!, \; K \geq 4$

Step-3: To prove $P(K+1)$ is true

$\cancel{2^{(K+1)} = (K+1)!}$

$\cancel{2^K 2^1 = (K+1)!}$

$(2^K) 2 < 2K!$

$2^K 2^1 < K!(K+1) \quad \{2 < K+1\}$

$2^{K+1} < (K+1)!$

$2^{K+1} < (K+1)!$

Which is $P(K+1)$

That is $P(K+1)$ is true whenever $P(K)$ is true

Problem-6 Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whether n is a positive integer.

Sol/n) Let $P(n)$: $n^3 - n$ is divisible by 3

Step-1: To prove $P(1)$ is true

For $\boxed{n=1}$. $1^3 - 1 = 0$ is divisible by 3

Hence $P(1)$ is true.

Step-2: Assume that $P(k)$ is true

ie, $(k^3 - k)$ is divisible by 3

Step-3: To prove $P(k+1)$ is true

i.e, $(k+1)^3 - (k+1)$ is divisible by 3

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$$

$$= k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$

$$= (k^3 - k) + 3k^2 + 3k$$

$$= (k^3 - k) + 3(k^2 + k)$$

which is divisible by 3

Hence $P(k+1)$ is true.

Chapter - 2.2

THE BASICS OF COUNTING

The Basic Counting Principles are

⇒ The Product Rule

⇒ The Sum Rule.

The Product rule:-

If one job can be done in 'm' ways and following this another job can be done in 'n' ways then the total no. of. ways in which both the jobs can be done in the stated order in "mn."

The Sum rule:-

If one job can be in 'm' ways and another job can be done in 'n' ways and if there is no way common to both jobs then the total no. of. ways in which either of the jobs can be done is Equal to m+n.

Example-① How many different bit strings of length seven are there?

Sol/n: Each of the seven bits can be chosen in two ways, because each bit is either 0 (or) 1.

∴ The Product rule shows there are a total of $2^7 = 128$ different bit string of length seven

Example-② How many different 2-digit numbers can be made from the digit 1, 2, 3, 4, 5, 6, 7, 8, 9, 0? When repetition is allowed? When repetition is not allowed?

Soln. Given that 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Case-(i) When repetition is allowed.

The tens places can be filled by 10 ways and the units places can be filled by 10 ways.

∴ The total no. of 2-digit numbers $= 10 \times 10$
$= 100$

Case-(ii) When repetition is not allowed

The tens place can be filled by 10 ways and the unit place can be filled by 9 ways

∴ The total no. of 2 digit numbers $= 10 \times 9$
$= 90$

Example-③ How many different 8 bit strings are there that begin and End with one.

Soln
A 8-bit string that begins and End with 1 can be constructed in 6 steps.

By selecting II bit, III bit, IV bit, V bit, VI bit VII bit and Each bit can be selected in 2 ways

Hence, the total no. of 8 bit strings that begins and end with 1 is Equal to

$$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$
$$= 2^6$$
$$= 64 //$$

Example - (10)

The chairs of an auditorium are to be ~~labeled~~ labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest no. of chairs that can be labeled differently?

Soln

Assigning to the seat one of the 26 - uppercase letter

Assigning to it one of the 100 possible integer

Using Product rule $= m \times n$
$= 26 \times 100 = 260 //$

THE PIGEONHOLE PRINCIPLE

Theorem: The Pigeonhole Principle

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is atleast one box containing two or more of the objects.

Proof: We prove this principle by the method of contradiction.

Suppose that none of the k boxes contains more than one object.

Hence the total no. of objects would be atmost k.

This is a contradiction

Since, there are atleast $k+1$ objects

Theorem: If 'n' pigeons are assigned to 'm' pigeonholes, and $m < n$, then atleast one pigeonhole contains two or more pigeons.

Theorem:- The generalization or Extension of the Pigeonhole Principle

If k Pigeons are assigned to n Pigeonholes, then one of the Pigeonhole must contain atleast $\left[\frac{k-1}{n}\right]+1$ Pigeons.

Example:- ① Show that among 13 children, there are atleast two children who were born in the same month.

Soln: Let us assume that 13 children as Pigeons and the 12 months (Jan, Feb, mar, - - - - - , Dec) as the Pigeonholes.

by the Pigeonhole Principle there will be atleast two children who were born in the same month.

Example-② Show that if any four numbers from 1 to 6 are chosen, then two of them will add to 7.

Soln: Let us form 3 sets containing two numbers whose sum is 7

ie, $A=(1,6)$, $B=(2,5)$, $C=(3,4)$

the four number that will be chosen to the set that contains it.

As there are only 3 sets, two numbers that there chosen is from the set whose sum is 7.

Example-(3) Show that if seven colours are used to paint 50 cars, atleast eight cars will have the same colour.

Soln:

Assume that 50 cars → Pigeons (k)

& 7 colour → Pigeonholes (n)

By Generalised Pigeonhole Principle, atleast

$$\left\lceil \frac{k-1}{n} \right\rceil + 1 = \left\lceil \frac{50-1}{7} \right\rceil + 1 = \left\lceil \frac{49}{7} \right\rceil + 1 = 7+1 = 8$$

∴ 8 cars will have the same colour.

Example-(4) Seven members of a family have total Rs-2,886 in their pockets. Show that atleast one of them must have atleast Rs-413 in his pocket.

Soln

Assume that members (7) → Pigeonholes (n)

& the Rupees (2,886) → Pigeons (k)

By Generalised Pigeonhole Principle, atleast

$$\left\lceil \frac{k-1}{n} \right\rceil + 1 = \cancel{\left\lceil \frac{2886}{7} \right\rceil + 1 =}$$

$$\left\lceil \frac{2886-1}{7} \right\rceil + 1 = \left\lceil \frac{2885}{7} \right\rceil + 1$$

$$= 412 + 1$$

$$= 413$$

Hence, there are 416 rupees in one member's pocket.

Example-5 If 9 books are to be kept it 4 shelves, there must be atleast one shelf which contain atleast 3 books.

Sol/n Let us assume book(9) → Pigeons(k)

& the shelves (4) → Pigeonholes (n)

By Extended Pigeonhole Principle.

$$\left[\frac{k-1}{n}\right]+1 = \left[\frac{9-1}{4}\right]+1 = 8/4 + 1 = 2+1 = 3$$

Hence there are 3 books in one shelf atleast

Example-6 How many people must you have to guarantee that atleast 9 of them will have birthday in the same day of the week.

Sol/n Let us assume the days week (7) → Pigeonholes.

& the ~~People~~ People → Pigeon (k)

Now 7 Pigeonholes and we've to find Pigeon

where n = 7, k = ?

By extended Pigeonhole Principle, atleast '9'

$$\cancel{\left[\frac{k-1}{n}\right] = 9} \qquad \cancel{\frac{k-1}{7} = 9 \Rightarrow k-1 =}$$

$$\left[\frac{k-1}{n}\right]+1 = 9$$

$$\left[\frac{k-1}{7}\right]+7 = 9$$

$$\left\lceil \frac{k-1}{7} \right\rceil + 1 = 9$$

$$\frac{k-1+7}{7} = 9$$

$$\frac{k+6}{7} = 9 \Rightarrow k+6 = 63 \Rightarrow k = 63-6, \boxed{k=57}$$

There must be 57 people to guarantee that atleast 9 of them will have birthdays in the same day of the week.

Chapter - 2.3

Permutation and Combinations

Permutation: nP_r (or) $P(n,r)$ (or) nP_r

A Permutation of a set of distinct objects is an ordered arrangement of these objects

ie, $nP_r = n(n-1)(n-2)(n-3)\cdots\cdots(n-r+1)$

$$\boxed{nP_r = \frac{n!}{(n-r)!}}$$

Result:

(i) $P(n,n) = n!$

(ii) $P(n,r) = 0$ if $r > n$

(iii) $P(n,0) = 1$.

Combinations: nC_r (or) $C(n,r)$

A Combination is a selection of objects without regard to order

$$\boxed{nC_r = \frac{n!}{r!(n-r)!}}$$

Result:

(i) $nC_n = nC_0 = 1$

(ii) $nC_r = nC_{n-r}$

(iii) $nC_r = \frac{nP_r}{r!}$

Problem - 1 Find the value of these quantities

$P(6,3)$, $P(8,1)$, $P(8,8)$, $C(5,3)$, $C(8,0)$

Soln -

Formula, $P(n,r) = \frac{n!}{(n-r)!}$

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

(i) $P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6\times5\times4\times\cancel{3}\times\cancel{2}}{\cancel{3}\times\cancel{2}} = 120$

(ii) $P(8,1) = \frac{8!}{(8-1)!} = \frac{8!}{7!} = \frac{8\times7\times6\times5\times4\times3\times2}{\cancel{7\times6\times5\times4\times3\times2}} = 8$

(iii) $P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = \frac{8\times7\times6\times5\times4\times3\times2}{1} = 40,320$

(iv) $C(5,3) = \frac{5!}{(5-3)!\,3!} = \frac{5!}{(2!)(3!)} = \frac{5\times\cancel{4}^{2}\times\cancel{3}\times\cancel{2}}{(\cancel{2}\times1)(\cancel{3\times2})} = 10$

(v) $C(8,0) = \frac{8!}{(8-0)!\,0!} = \frac{\cancel{8!}}{\cancel{8!}} = 1$

Problem - 2 Determine the value of n if $4({}^{n}P_3) = {}^{(n+1)}P_3$

Soln

Given $4({}^{n}P_3) = {}^{(n+1)}P_3$

$$4\left[\frac{n!}{(n-3)!}\right] = \frac{(n+1)!}{(n+1-3)!}$$

$$4\left[\frac{n!}{(n-3)!}\right] = \cancel{(n+1)\,n!}\ \frac{n!\,(n+1)}{(n-2)!}$$

$$4\left[\frac{n!}{(n-3)!}\right] = \frac{n!\,(n+1)}{(n-3)!\,(n-2)}$$

$$4\left[\cancel{\frac{n!}{(n-3)!}}\right] = \left[\cancel{\frac{n!}{(n-3)!}}\right]\left[\frac{n+1}{n-2}\right]$$

$$4 = \frac{n+1}{n-2}$$

$$4(n-2) = n+1$$

$$4n-8 = n+1$$

$$4n-n = 1+8$$

$$3n = 9$$

$$n = 9/3 \implies \boxed{n=3}$$

Problem-③ Determine the value of n of $20C_{n+2} = 20C_{2n-1}$

Soln Given that $20C_{n+2} = 20C_{2n-1}$

W.k.T $nC_x = nC_y \implies n = x+y$ (or) $x=y$

$$\therefore\ n+2 = 2n-1$$

$$2+1 = 2n-n$$

$$3 = n$$

$$\boxed{n=3}$$

(4) How many possibilities are there for the win, place and show [first, second and third] position in a horse race with 12 horses if all order of finish are possible

Soln

The no. of ways to pick the three winners is the no. of ordered selections of three elements from 12

ie, $P(12,3) = 12 \times 11 \times 10 = 1320.$

(5) How many permutations of $\{a, b, c, d, e, f, g\}$

(i) End with a, (ii) begin with c, (iii) begin with c and end with a, (iv) c and a occupy the end places.

Soln Given that $\{a, b, c, d, e, f, g\}$.

Total no. of letters $= 7$.

(i) The last position can be filled in only one way

The remaining 6 letters can be arranged in $6!$ ways

$\therefore$ The total no. of permutations ending with 'a' are

$= (6!)(1) = 720$

(ii) The first position can be filled in only one way

The remaining 6 letters can be arranged in $6!$ ways

The total no. of permutations starting with 'c' are

$= (1)(6!) = 720$

(iii) The first and last Position can be filled by only one way

The remaining letters can be arranged in $(5!)$ ways

The total no. of Permutations begin with 'c' and End with 'a' $= (1)(5!)(1) = 120$.

(iv) 'c' and 'a' occupy End Positions in $(2!)$ ways and the remaining 5 letters can be arranged in $(5!)$ ways.

The total no. of Permutations $= (5!)(2!)$
$= (120)(2) = 240$.

⑥ How many Permutations of the letters ABCDEFG Contain (i) the string BCD, (ii) the string CFGA, (iii) the string BA and GF, (iv) the string ABC and DE, (v) the string ABC and CDE.

Soln:- Given that the letters A B C D E F G.

(i) Taking "BCD" as one object, we have the following 5 objects : A, (BCD), E, F, G

The no. of ways $= 5! = 120$ ways

(ii) Taking "CFGA" as one object, we have the following 4 objects : (CFGA), B, D, E

The no. of ways $= 4! = 24$ ways

(iii) The objects (BA), (GF), C, D, E can be permuted in $5! = 120$ ways

(iv) The objects (ABC), (DE), F, G can be permuted in $4! = 24$ ways

(v) Eventhough (ABC) and (CDE) are two strings. But they have common letter C. So, we can write in another way (ABCDE).

Hence, we have (ABCDE), F, G can be permuted in $3! = 6$ ways.

⑦ How many bit string of length 10 contain

(i) Exactly four 1's, (ii) Atmost four 1's

(iii) Atleast four 1's (iv) an Equal number of 0's and 1's

Soln:

A bit string of length 10 can be considered.

(i) The bit string have Exactly four 1's, we have four 1's & six 0's

No. of required bit string $= \frac{10!}{4!\,6!} = 210$

(ii) The bit string have at most four 1's.

Required no. of bit string

$= \frac{10!}{0!\,10!} + \frac{10!}{1!\,9!} + \frac{10!}{2!\,8!} + \frac{10!}{3!\,7!} + \frac{10!}{4!\,6!}$

$= 1 + 10 + 45 + 120 + 210$

$= 386$

(iii) The bit string have atleast four 1's

Required no. of bit string

$$= \frac{10!}{4!6!} + \frac{10!}{5!5!} + \frac{10!}{6!4!} + \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!0!}$$

$$= 210 + 252 + 210 + 120 + 45 + 10 + 1$$

$$= 848$$

(iv) The bit string have an Equal number of 0's and 1's

Required no. of bit string

$$= \frac{10!}{5!5!} = \frac{3628800}{14,400} = 252$$

(8) Determine n. if $P(n,2) = 72$.

Soln

Given that, $P(n,2) = 72$

W.k.T $P(n,r) = \frac{n!}{(n-r)!}$

$$\frac{n!}{(n-2)!} = 72$$

$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 72$$

$$n(n-1) = 72$$

$$n^2 - n = 72$$

$$n^2 - n - 72 = 0$$

$$(n+8)(n-9) = 0$$

$n+8=0$ | $n-9=0$

$\boxed{n=-8}$ | $\boxed{n=9}$

'n' must be Positive

So, $\boxed{n=9}$

(9) A team of 11 Players is to be choosen from 15 members. In how many ways can this be done if

(i) one Particular Player is always included

(ii) Two such Players have always to be included

Solu: Given that, Total. no. of members $= 15$

& a team have $= 11$ Players

(i) one particular Player is always included.

∴ one Player is fixed, Possible 14 from 10

Required. no. of ways $= 14C_{10} = 1,001$

(ii) Two such Players have always to be included

Two Players is fixed, Possible ways 13 from 9.

Required. no. of ways $= 13C_9 = 715$.

(10) There are 6 men and 5 women in a room, Find the members of ways 4 Person can be drawn from the room if (i) they can be male (or) Female. (ii) 2 must be men and 2 women. (iii) they must all are of the same sex.

Solu: Given that, 6-men & 5-women,

Total. no-of. members $= 11$.

4-Person can be drawn from 11.

(i) Required no. of ways $= 11C_4 =$

(ii) Required no. of ways $= \dfrac{11C_4}{6C_2 \times 5C_2} = \dfrac{330}{15 \times 10} = \dfrac{330}{150} = \dfrac{33}{15}$

(iii) Required no. of ways $= 6C_4$ (or) $5C_4$

$= 6C_4 + 5C_4 = 15 + 5 = 20$

(11) From a Committe Consisting of 6-men and 7-women in how many ways can be select a Committee of
(i) 3 men and 4 women, (ii) 4 members which has atleast one women, (iii) 4 Person that has atmost one men
(iv) 4 Person of both sex.

Soln:- Given that, 6-men and 7-women; Total members $= 6+7$
$= 13$

(i) 3 men and 4 women

Required no. of ways $= 6C_3 \times 7C_4$

(ii) 4 members which has atleast one women

~~Required~~ Possible ways $= (7C_1 \times 6C_3)$ (or) $(7C_2 \times 6C_2)$ (or) $(7C_3 \times 6C_1)$ (or) $(7C_4 \times 6C_0)$

Required no. of ways $= (7C_1 \times 6C_3) + (7C_2 \times 6C_2) + (7C_3 \times 6C_1) + (7C_4 \times 6C_0)$

$= 140 + 315 + 210 + 35$

$= 700.$

(iii) A person that has atmost one men.

Possible ways $= (6C_1 \times 7C_3)$ (or) $(6C_0 \times 7C_4)$

Required no. of ways $= (6C_1 \times 7C_3) + (6C_0 \times 7C_4)$

$= 210 + 35$

$= 245$

iv) 4 person of both sex.

Possible ways $=$ (1m, 3w) (or) (2m, 2w) (or) (3m, 1w)

Required no. of ways $= (6C_1 \times 7C_3) + (6C_2 \times 7C_2) + (6C_3 \times 7C_1)$

$= 210 + 315 + 140$

$= 665$

(12) How many permutation can be made out of the letters of the word BASIC? How many of there (i) Begin with B, (ii) End with C, (iii) B and C occupy the End place.

Soln: Given that, BASIC → all are different letters.

Total no. of letters $= 5$

(i) Begin with B letter.

Required no. of ways $= (1)\ 4P_4 = (1)(4!) = 24$

(ii) End with C letters.

Required no. of ways $= (4P_4)(1) = (4!)(1) = 24$

(iii) B and C occupy the End place.

Required no. of ways $= (3P_3)(2P_2) = (3!)(2!) = 12$

(13) How many bit string of length 12 contain Exactly four 1's.

Soln

Given that, bit string of length is 12.

bit means - 0 & 1

we need Exactly four 1's means (4 - 1's & 8 - 0's)

Required no. of ways $= \dfrac{12!}{4!\,8!}$

(14) How many different words are there in the word.

(i) COMPUTER, (ii) MATHEMATICS, (iii) ENGINEERING.

Soln

(i) Given that "COMPUTER"

Total no. of letters = 8 (all are different)

Required no. of ways $= 8P_8 = 8! = 40,320$

(ii) Given that "MATHEMATICS"

Total no. of letters = 11

Here, M - occurs = 2 times

A - occurs = 2 times

T - occurs = 2 times

H, E, I, C, S - occurs = 1 times

$$\text{Required. no. of. ways} = \frac{11!}{(2!)(2!)(2!)} = \frac{3,99,16,800}{(4)(4)(4)}$$

$$= \frac{3,99,16,800}{64} = 623,700$$

(iii) Given that "ENGINEERING"

Total. no. of. letters = 11

Here, E - occurs = 3 (times)
N - occurs = 3 (times)
G - occurs = 2 times
I - occurs = 2 times
R - occurs = 1 times

$$\text{Required. no. of. ways} = \frac{11!}{(3!)(3!)(2!)(2!)} = \frac{39,916,800}{(6)(6)(4)(4)}$$

$$= \frac{3,99,16,800}{576} = 69,300$$

(15) How many ways are there to assign 5 different job to 4 different Employee's, if Every Employee's assigned atleast one job.

Soln Given that 5-different job & 4-different Employee's

$$\text{Required. no. of. ways} = (5C_2 \times 3P_2) + (5C_2 \times 3P_2) + (5C_2 \times 3P_2) + (5C_2 \times 3P_2)$$

$$= 4 \times [5C_2 \times 3P_2]$$

$$= 4 \times [10 \times 6] = 4[60]$$

$$= 240.$$

(16) In How many different ways can 5-men & 5-women sit arround the table.

Soln Given that 5-men & 5-women

$$\text{Required. no. of. ways} = 10P_{10} = 10!$$

Chapter - 2.4

RECURRENCE RELATIONS

Defn: A recurrence relation for the sequence $\{a_n\}$ is an equation that show a_n in terms of one or more of the previous terms of the sequence $a_0, a_1, a_2, \ldots, a_{n-1}, \ldots$, for all integers n, with $n \geq n_0$, where n_0 is a non-negative integer.

① Let $\{A_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-2} + a_{n-1}$ for $n = 2, 3, 4, 5, \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Soln: Given that $a_n = a_{n-2} + a_{n-1}$ / $a_0 = 3$, $a_1 = 5$

To find:

(i) $a_2 = a_{2-2} + a_{2-1}$

$= a_0 + a_1$

$a_2 = 3 + 5$

$\boxed{a_2 = 8}$

(ii) $a_3 = a_{3-2} + a_{3-1}$

$= a_1 + a_2$

$= 5 + 8$

$\boxed{a_3 = 13}$

(2) Find the first five terms of the sequence defined by Each of these recurrence relations and initial Conditions

(i) $a_n = 6a_{n-1}$, given $a_0 = 2$

(ii) $a_n = a_{n-1} + 3a_{n-2}$, given $a_0 = 1, a_1 = 2$

Soln:

(i) Given that $a_n = 6a_{n-1}$ / $a_0 = 2$

$$a_1 = 6a_{1-1} = 6a_0 = 6(2) = 12$$

$$a_2 = 6a_{2-1} = 6a_1 = 6(12) = 72$$

$$a_3 = 6a_{3-1} = 6a_2 = 6(72) = 432$$

$$a_4 = 6a_{4-1} = 6a_3 = 6(432) = 2592$$

$$a_5 = 6a_{5-1} = 6a_4 = 6(2592) = 15{,}552.$$

(ii) Given that $a_n = a_{n-1} + 3a_{n-2}$ / $a_0 = 1$, $a_1 = 2$

$$a_2 = a_{2-1} + 3a_{2-2} = a_1 + 3a_0 = 2 + 3(1) = 2+3 = 5$$

$$a_3 = a_{3-1} + 3a_{3-2} = a_2 + 3a_1 = 5 + 3(2) = 5 + 6 = 11$$

$$a_4 = a_{4-1} + 3a_{4-2} = a_3 + 3a_2 = 11 + 3(5) = 11 + 15 = 26$$

$$a_5 = a_{5-1} + 3a_{5-2} = a_4 + 3a_3 = 26 + 3(11) = 26 + 33 = 59$$

3) Let $a_n = 2^n + 5(3^n)$. for $n = 0, 1, 2, 3, \ldots$

(i) Find a_0, a_1, a_2, (ii) Show that $a_4 = 5a_3 - 6a_2$

Soln

(i) Given that $a_n = 2^n + 5(3^n)$ / for $n = 0, 1, 2, 3 \ldots$

$$a_0 = 2^0 + 5(3^0) = 1 + 5(1) = 1 + 5 = 6$$

$$a_1 = 2^1 + 5(3^1) = 2 + 5(3) = 2 + 15 = 17$$

$$a_2 = 2^2 + 5(3^2) = 4 + 5(9) = 4 + 45 = 49$$

(ii) Given that, show that $a_4 = 5a_3 - 6a_2$

L.H.S $a_4 = 2^4 + 5(3^4) = 16 + 5(81)$

$$= 16 + 405$$

$$\boxed{a_4 = 421}$$

R.H.S $5a_3 - 6a_2$

To find $a_3 = 2^3 + 5(3^3) = 8 + 5(27) = 8 + 135 = 143$

$$= 5a_3 - 6a_2$$

$$= 5(143) - 6(49)$$

$$= 715 - 294$$

$$\boxed{5a_3 - 6a_2 = 421}$$

L.H.S = R.H.S

$$\therefore \boxed{a_4 = 5a_3 - 6a_2}$$

(4) Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$. if

(i) $a_n = 3(-1)^n + 2^n - n + 2$

(ii) $a_n = 7(2)^n - n + 2$.

Soln:- Given that $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$

(i) $a_n = 3(-1)^n + 2^n - n + 2$

step-1 $a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$

$= 3(-1)^n(-1) + 2^n(2^{-1}) - n + 1 + 2$

$= -3(-1)^n + 2^n(1/2) - n + 3$

$a_{n-1} = -3(-1)^n + 1/2(2)^n - n + 3 \longrightarrow (1)$

step-2 $a_{n-2} = 3(-1)^{n-2} + (2)^{n-2} - (n-2) + 2$

$= 3(-1)^n(-1)^{-2} + (2)^n(2)^{-2} - n + 2 + 2$

$= 3(-1)^n(1) + (2)^n(1/2^2) - n + 4$

$= 3(-1)^n + (2)^n(1/4) - n + 4$

$a_{n-2} = 3(-1)^n + 1/4(2)^n - n + 4$

$2(a_{n-2}) = 3(2)(-1)^n + \frac{\cancel{2}}{2\cancel{4}}(2)^n - 2n + 8$

$2(a_{n-2}) = 6(-1)^n + 1/2(2)^n - 2n + 8 \longrightarrow (2)$

$\therefore \quad a_{n-1} + 2a_{n-2} + 2n - 9$, {using ① & ②}

$= -3(-1)^n + \frac{1}{2}(2)^n - n + 3 + 6(-1)^n + \frac{1}{2}(2)^n - 2n + 8 + 2n - 9$

$= -3(-1)^n + 6(-1)^n + \frac{1}{2}(2)^n + \frac{1}{2}(2)^n - 3n + 2n + 3 + 8 - 9$

$= 3(-1)^n + (2)^n - n + 2$

$= a_n. \implies \boxed{a_{n-1} + 2a_{n-2} + 2n - 9 = a_n}$

(ii) Given that $a_n = 7(2)^n - n + 2$

Step-1 $\quad a_{n-1} = 7(2)^{n-1} - (n-1) + 2$

$= 7(2)^n 2^{-1} - n + 1 + 2$

$= 7(2)^n (\frac{1}{2}) - n + 3$

$a_{n-1} = \frac{7}{2}(2)^n - n + 3 \longrightarrow ①$

Step-2 $\quad a_{n-2} = 7(2)^{n-2} - (n-2) + 2$

$= 7(2)^n (2)^{-2} - n + 2 + 2$

$= 7(2)^n (\frac{1}{2^2}) - n + 4$

$= 7(2)^n (\frac{1}{4}) - n + 4$

$a_{n-2} = \frac{7}{4}(2)^n - n + 4$

$2(a_{n-2}) = \frac{7 \cancel{\times 4}}{\cancel{4}\, 2}(2)^n - 2n + 8$

$2(a_{n-2}) = \frac{7}{2}(2)^n - 2n + 8 \longrightarrow ②$

$\therefore\ a_{n-1} + 2a_{n-2} + 2n - 9.$ {using (1) & (2)}

$= \frac{7}{2}(2)^n - n + 3 + \frac{7}{2}(2)^n - 2n + 8 + 2n - 9$

$= \frac{7}{2}(2)^n + \frac{7}{2}(2)^n - n - \cancel{2n} + \cancel{2n} + 3 + 8 - 9$

$= \frac{\cancel{14}^{7}}{\cancel{2}}(2)^n - n + 2$

$= 7(2)^n - n + 2$

$= a_n.$

$\therefore\ \boxed{a_{n-2} + 2a_{n-2} + 29 - 9 = a_n}$

SOLVING LINEAR RECURRENCE RELATIONS

Defn:

A linear recurrence relation with constant coefficient is of the form.

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{k-1} = f(n)$$

where c is are constants.

A linear homogenous recurrence relation with constant coefficients of degree k is of the form.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, c_3, \ldots c_k$ are real number, & $c_k \neq 0$.

The three methods of solving recurrence relation are (i) Iteration, (ii) characteristic roots (iii) Generating function.

Example -① What is the solution of recurrence relation?

$a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

Soln:

Given that $a_n = 5a_{n-1} - 6a_{n-2}$ → ① | $a_0 = 1$, $a_1 = 0$

Let $a_n = r^n$ be a solution of the given Equation.

$$① \Longrightarrow a_n - 5a_{n-1} + 6a_{n-2} = 0$$

Put $\boxed{a_n = r^n}$

$$r^n - 5r^{n-1} + 6r^{n-2} = 0$$

$$r^n [1 - 5r^{-1} + 6r^{-2}] = 0$$

$$r^n\left[1-\frac{5}{r}+\frac{6}{r^2}\right]=0$$

$$r^n\left[\frac{r^2-5r+6}{r^2}\right]=0$$

$r^2-5r+6=0$ is the characteristic Equ

$(r-3)(r-2)=0$

$r-3=0$ | $r-2=0$

$\boxed{r=3}$ | $\boxed{r=2}$

$\boxed{r_1=3}$ | $\boxed{r_2=2}$

∴ By the theorem $a_n=\alpha(r_1)^n+\beta(r_2)^n$

$$a_n=\alpha(3)^n+\beta(2)^n \longrightarrow (2)$$

Given that $\boxed{a_0=1}$ $\Rightarrow$ $\boxed{n=0}$ in Equ (2)

$$a_0=\alpha(3)^0+\beta(2)^0$$

$$1=\alpha(1)+\beta(1)$$

$$1=\alpha+\beta$$

$$\boxed{\alpha+\beta=1} \longrightarrow (3)$$

$\boxed{a_1=0}$ $\Rightarrow$ $\boxed{n=1}$ in Equ (2)

$$a_1=\alpha(3)^1+\beta(2)^1$$

$$0=\alpha 3+\beta(2)$$

$$0=3\alpha+2\beta$$

$$\boxed{3\alpha+2\beta=0} \longrightarrow (4)$$

from (3) & (4)

$(3) \times 2 \Rightarrow \quad 2\alpha + \cancel{2\beta} = 2$

$(4) \Longrightarrow \quad 3\alpha + \cancel{2\beta} = 0$

$\quad\quad (-) \quad (-) \quad (-)$

$-\alpha = 2 \Rightarrow \boxed{\alpha = -2}$

Put $\boxed{\alpha = -2}$ in Equ (3)

$(3) \Rightarrow \alpha + \beta = 1$

$-2 + \beta = 1$

$\beta = 1 + 2$

$\boxed{\beta = 3}$

Put $\boxed{\alpha = -2}$ & $\boxed{\beta = 3}$ in Equ (2)

$(2) \Rightarrow a_n = \alpha(3)^n + \beta(2)^n$

$= (-2)(3)^n + 3(2)^n$

$$\boxed{a_n = 3(2)^n - 2(3)^n}$$

Example-(2) What is the solution of the recurrence relation $a_n = 2a_{n-1}$, for $n \geq 1$, $a_0 = 3$

Soln:

Given that $\underbrace{a_n = 2a_{n-1}}_{\to (1)}$, $n \geq 1$, $a_0 = 3$

Let $\boxed{a_n = r^n}$

$(1) \Rightarrow a_n = 2a_{n-1}$

$a_n - 2a_{n-1} = 0$

$r^n - 2(r)^{n-1} = 0$

$$r^n - 2(r)^{n-1} = 0$$

$$r^n - 2(r^n)\, r^{-1} = 0$$

$$r^n - 2r^n(1/r) = 0$$

$$r^n\left[1 - 2/r\right] = 0$$

$$r^n\left[\frac{r-2}{r}\right] = 0$$

$r - 2 = 0$ is a characteristic Equ.

$\boxed{r = 2} \Rightarrow \boxed{r_1 = 2}$

By the theorem $\boxed{a_n = \alpha(r_1)^n} \Rightarrow \boxed{a_n = \alpha(2)^n} \rightarrow (2)$

Given that $\boxed{a_0 = 3} \Rightarrow \boxed{n = 0}$ in Equ (2)

$$a_0 = \alpha(2)^0$$

$$3 = \alpha(1)$$

$$3 = \alpha \Rightarrow \boxed{\alpha = 3}$$

(2) $\Rightarrow a_n = \alpha(2)^n$

$$\boxed{a_n = 3(2)^n}$$

(3) Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$; $a_1 = -4$; $a_2 = 8$.

Solu:

Given that $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ $\to$ (1) / $a_0 = 7$, $a_1 = -4$, $a_2 = 8$.

let $\boxed{a_n = r^n}$

$$(1) \Rightarrow a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$$

$$a_n - 2a_{n-1} - 5a_{n-2} + 6a_{n-3} = 0$$

$$r^n - 2(r)^{n-1} - 5(r)^{n-2} + 6(r)^{n-3} = 0$$

$$r^n - 2r^n r^{-1} - 5r^n r^{-2} + 6r^n r^{-3} = 0$$

$$r^n - 2r^n \tfrac{1}{r} - 5r^n \tfrac{1}{r^2} + 6r^n \tfrac{1}{r^3} = 0$$

$$r^n\left[1 - \tfrac{2}{r} - \tfrac{5}{r^2} + \tfrac{6}{r^3}\right] = 0$$

$$r^n\left[\frac{r^3 - 2r^2 - 5r + 6}{r^3}\right] = 0$$

$\therefore$ $r^3 - 2r^2 - 5r + 6 = 0$ is a characteristic Equn. $\to$ (2)

Here, $\boxed{r=1}$ is a root.

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & 0 & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$$\Rightarrow r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$\begin{array}{c|c} \multicolumn{2}{c}{-6} \\ -3 & 2 \\ \hline \multicolumn{2}{c}{-1} \end{array}$$

$(r-3)(r+2)=0$

$r-3=0 \quad | \quad r+2=0$

$\boxed{r=3} \quad | \quad \boxed{r=-2}$

The roots are, $\boxed{r_1=1}$; $\boxed{r_2=3}$; $\boxed{r_3=-2}$

By the Recurrence theorem,

$$a_n = \alpha(r_1)^n + \beta(r_2)^n + \gamma(r_3)^n$$

$$a_n = \alpha(1)^n + \beta(3)^n + \gamma(-2)^n \longrightarrow (2)$$

Given that (i) $\boxed{a_0=7} \Rightarrow \boxed{n=0}$ in Eqn (2)

$$a_0 = \alpha(1)^0 + \beta(3)^0 + \gamma(-2)^0$$

$$7 = \alpha+\beta+\gamma \longrightarrow$$

$$\boxed{\alpha+\beta+\gamma=7} \longrightarrow (3)$$

(ii) $\boxed{a_1=-4} \Rightarrow \boxed{n=1}$ in Eqn (2)

$$a_1 = \alpha(1)^1 + \beta(3)^1 + \gamma(-2)^1$$

$$-4 = \alpha + 3\beta - 2\gamma$$

$$\boxed{\alpha+3\beta-2\gamma=-4} \longrightarrow (4)$$

(iii) $\boxed{a_2=8} \Rightarrow \boxed{n=2}$ in Eqn (2)

$$a_2 = \alpha(1)^2 + \beta(3)^2 + \gamma(-2)^2$$

$$8 = \alpha(1) + \beta(9) + \gamma(4)$$

$$8 = \alpha + 9\beta + 4\gamma$$

$$\boxed{\alpha+9\beta+4\gamma=8} \longrightarrow (5)$$

By Solving (3), (4) & (5)

$$(3) - (4) \Rightarrow \quad \cancel{\alpha} + \beta + \gamma = 7$$
$$\cancel{\alpha} + 3\beta - 2\gamma = -4$$
$$(-) \quad (-) \quad (+) \quad (+)$$
$$-2\beta + 3\gamma = 11 \rightarrow (6)$$

$$(4) - (5) \Rightarrow \quad \cancel{\alpha} + 3\beta - 2\gamma = -4$$
$$\cancel{\alpha} + 9\beta + 4\gamma = 8$$
$$(-) \quad (-) \quad (-) \quad (-)$$
$$-6\beta - 6\gamma = -12 \rightarrow (7)$$

$$(7) \Rightarrow -6\beta - \cancel{6\gamma} = -12$$
$$(6) \times 2 \Rightarrow -4\beta + \cancel{6\gamma} = 22$$
$$-10\beta = 10$$
$$-\beta = 1$$
$$\boxed{\beta = -1} \rightarrow (8)$$

Use (8) in (6)

$$(6) \Rightarrow -2\beta + 3\gamma = 11$$
$$-2(-1) + 3\gamma = 11$$
$$+2 + 3\gamma = 11$$
$$3\gamma = 11 - 2$$
$$3\gamma = 9$$
$$\gamma = 9/3$$
$$\boxed{\gamma = 3} \rightarrow (9)$$

Use (8) & (9) in (3)

$$(3) \Rightarrow \alpha + \beta + \gamma = 7$$
$$\alpha - 1 + 3 = 7$$
$$\alpha + 2 = 7$$
$$\alpha = 7 - 2$$
$$\boxed{\alpha = 5} \rightarrow (10)$$

Put $\boxed{\alpha=5}$, $\boxed{\beta=-1}$ & $\boxed{\gamma=3}$ in Eqn (2)

(2) $\Rightarrow$ $a_n = \alpha(1)^n + \beta(3)^n + \gamma(-2)^n$

$$a_n = 5(1)^n - 1(3)^n + 3(-2)^n$$

(4) Solve the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ for $n \geq 2$, $a_0 = 16$; $a_1 = 80$.

Soln: Given that $a_n = 8a_{n-1} - 16a_{n-2}$ → (1) | $n \geq 2$, $a_0 = 16$, $a_1 = 80$

Let $\boxed{a_n = r^n}$

(1) $\Rightarrow$ $a_n = 8a_{n-1} - 16a_{n-2}$

$$a_n - 8a_{n-1} + 16a_{n-2} = 0$$

$$r^n - 8r^{n-1} + 16r^{n-2} = 0$$

$$r^n - 8r^n r^{-1} + 16r^n r^{-2} = 0$$

$$r^n - 8r^n \tfrac{1}{r} + 16r^n \tfrac{1}{r^2} = 0$$

$$r^n\left[1 - \frac{8}{r} + \frac{16}{r^2}\right] = 0$$

$$r^n\left[\frac{r^2 - 8r + 16}{r^2}\right] = 0$$

$\therefore$ $r^2 - 8r + 16 = 0$ is the characteristic Eqn.

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ii) $r^2 - 8r + 16 = 0$

$$\begin{array}{c} 16 \\ -4 \mid -4 \\ 8 \end{array}$$

$(r-4)(r-4) = 0$

$r-4=0 \;|\; r-4=0$

$\boxed{r_1=4} \;|\; \boxed{r_2=4}$

$\boxed{r=4}$ is Repeated Root.

By the recurrence theorem,

$$a_n = (\alpha + n\beta) r^n$$

$$a_n = (\alpha + n\beta)(4)^n \longrightarrow 2$$

Given that initial value

(i) $\boxed{a_2 = 16} \Rightarrow \boxed{n=2}$.

$2 \Rightarrow a_2 = (\alpha + 2\beta)(4)^2$

$16 = (\alpha + 2\beta)(16)$

$16 = 16\alpha + 32\beta$

$16\alpha + 32\beta = 16 \longrightarrow 3$

(ii) $\boxed{a_3 = 80} \Rightarrow \boxed{n=3}$

$2 \Rightarrow a_3 = (\alpha + 3\beta)(4)^3$

$80 = (\alpha + 3\beta)(64)$

$80 = 64\alpha + 192\beta$

$64\alpha + 192\beta = 80 \longrightarrow 4$

from 3 & 4

$4 \Rightarrow \cancel{64\alpha} + 192\beta = 80$

$3 \times 4 \Rightarrow \cancel{64\alpha} + 128\beta = 64$

$(-) \quad (-) \quad (-)$

$64\beta = 16$

$64\beta = 16$

$\beta = 16/64 \Rightarrow \boxed{\beta = 1/4}$

Put $\beta = 1/4$ in Eqn (3)

(3) $\Rightarrow 16\alpha + 32\beta = 16$

$16\alpha + \overset{8}{\cancel{32}}(1/\cancel{4}) = 16$

$16\alpha + 8 = 16$

$16\alpha = 16 - 8$

$16\alpha = 8$

$\alpha = 8/16 \Rightarrow \boxed{\alpha = 1/2}$

Put the value $\alpha = 1/2$ & $\beta = 1/4$ in Eqn (2)

(2) $\Rightarrow a_n = (\alpha + n\beta)(4)^n$

$a_n = \left[1/2 + n(1/4)\right](4)^n$ //

(5) Solve $S(n+2) - 5S(n+1) + 6S(n) = 0$, for $n \geq 0$ with $S(0) = 1$, $S(1) = 1$

Soln

Given that $S(n+2) - 5S(n+1) + 6S(n) = 0$ | $S(0) = 1$, $S(1) = 1$

ie, $a_{n+2} - 5a_{n+1} + 6a_n = 0 \rightarrow (1)$ | $a_0 = 1$, $a_1 = 1$

we rewrite the Eqns (1)

$r^2 - 5r + 6 = 0$ is the char Eqns

$$r^2-5r+6=0$$

$$\begin{array}{c} 6 \\ -3 \mid -2 \\ \hline -5 \end{array}$$

$$(r-3)(r-2)=0$$

$$r-3=0 \quad\Big|\quad r-2=0$$

$$\boxed{r=3} \quad\Big|\quad \boxed{r=2}$$

$$\boxed{r_1=3} \quad\Big|\quad \boxed{r_2=2}$$

By the recurrence theorem,

$$a_n = \alpha(r_1)^n + \beta(r_2)^n$$

$$a_n = \alpha(3)^n + \beta(2)^n \longrightarrow (2)$$

Given initial value

(i) $\boxed{a_0=1} \Rightarrow \boxed{n=0}$ in Eqn (2)

$$(2) \Rightarrow a_0 = \alpha(3)^0 + \beta(2)^0$$

$$1 = \alpha(1) + \beta(1)$$

$$1 = \alpha + \beta$$

$$\alpha + \beta = 1 \longrightarrow (3)$$

(ii) $\boxed{a_1=1} \Rightarrow \boxed{n=1}$ in Eqn (2)

$$(2) \Rightarrow a_1 = \alpha(3)^1 + \beta(2)^1$$

$$1 = \alpha(3) + \beta(2)$$

$$1 = 3\alpha + 2\beta$$

$$3\alpha + 2\beta = 1 \longrightarrow (4)$$

from (3) & (4)

(4) $\Rightarrow$ $3\alpha + 2\beta = 1$

(3) $\times 2 \Rightarrow$ $2\alpha + 2\beta = 2$

$\underset{(-)}{} \quad \underset{(-)}{} \quad \underset{(-)}{}$

$\boxed{\alpha = -1}$

Put $\alpha = -1$ in Eqn (3)

$\alpha + \beta = 1$

$-1 + \beta = 1 \Rightarrow \beta = 1 + 1 \Rightarrow \boxed{\beta = 2}$

Put the value of $\boxed{\alpha = -1}$ & $\boxed{\beta = 2}$ in Eqn (2)

(2) $\Rightarrow$ $a_n = \alpha(3)^n + \beta(2)^n$

$= -1(3)^n + 2(2)^n$

$a_n = 2(2)^n - (3)^n$ //

Chapter - 2.5

GENERATING FUNCTIONS

Defn:

The generating function for the sequence $a_0, a_1, a_2, \ldots\ldots a_k, \ldots$ of real numbers is the finite series.

$$G(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n + \cdots$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Note:

① $\sum_{n=0}^{\infty} a_n x^n = G(x)$

② $\sum_{n=1}^{\infty} a_n x^n = G(x) - a_0$

③ $\sum_{n=2}^{\infty} a_n x^n = G(x) - a_0 - a_1x$

In General terms of L.H.S

$$\sum_{n=i}^{\infty} a_{n-j}\, x^n = \left[G(x) - a_0 - a_1x - a_2x^2 - \cdots\right] x^j$$

Case-(i) $\boxed{n \geq 2 \text{ for } n-\text{terms}}$

$$\sum_{n=2}^{\infty} a_n x^n = \left[G(x) - a_0 - a_1x\right] x^0$$

$$\sum_{n=2}^{\infty} a_{n-1} x^n = \left[G(x) - a_0\right] x^1$$

$$\sum_{n=2}^{\infty} a_{n-2} x^n = \left[G(x)\right] x^2$$

Case-(ii) $\boxed{n \geq 2 \text{ for } n+\text{terms}}$

$$\sum_{n=2}^{\infty} a_n x^n = \left[G(x) - a_0 - a_1x\right] x^0$$

$$\sum_{n=2}^{\infty} a_{n+1} x^n = \left[G(x) - a_0 - a_1x - a_2x^2\right] x^{-1}$$

Case-(iii) $\boxed{n \geq 1, \text{ for } n\text{-terms}}$

$$\sum_{n=1}^{\infty} a_n x^n = [G(x) - a_0]\, x^0$$

$$\sum_{n=1}^{\infty} a_{n-1} x^n = [G(x)]\, x^1$$

Case-(iv) $\boxed{n \geq 1, \text{ for } n+\text{ terms}}$

$$\sum_{n=1}^{\infty} a_n x^n = [G(x) - a_0]\, x^0$$

$$\sum_{n=1}^{\infty} a_{n+1} x^n = [G(x)]\, x^{-1}$$

Case-(iv) $\boxed{n \geq 0, \text{ for } n+\text{ terms}}$

$$\sum_{n=0}^{\infty} a_n x^n = [G(x)]\, x^0$$

$$\sum_{n=0}^{\infty} a_{n+1} x^n = [G(x) - a_0]\, x^{-1}$$

$$\sum_{n=0}^{\infty} a_{n+2} x^n = [G(x) - a_0 - a_1 x]\, x^{-2}$$

R.H.S terms

I - $\boxed{\text{for } a^n x^n \text{ terms}}$

① $$\sum_{n=0}^{\infty} a^n x^n = \frac{1}{1-ax}$$

② $$\sum_{n=1}^{\infty} a^n x^n = \frac{1}{1-ax} - 1$$

③ $$\sum_{n=2}^{\infty} a^n x^n = \frac{1}{1-ax} - 1 - ax$$

II - $\boxed{\text{for } n x^n \text{ terms}}$

① $$\sum_{n=0}^{\infty} n x^n \text{ (or) } \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

② $$\sum_{n=2}^{\infty} n x^n = \left[\frac{x}{(1-x)^2}\right] - x.$$

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(1) Solve the recurrence relation $a_n + 3a_{n-1} - 4a_{n-2} = 0, n \geq 2$ with $a_0 = 3; a_1 = -2$ using generating function.

Soln:

Given that $a_n + 3a_{n-1} - 4a_{n-2} = 0$, | $n \geq 2$, $a_0 = 3, a_1 = -2$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ is generating function

multiply by $\sum_{n=2}^{\infty} x^n$ on both side,

$$\sum_{n=2}^{\infty} a_n x^n + 3\sum_{n=2}^{\infty} a_{n-1} x^n - 4\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[G(x) - a_0 - a_1 x]x^0 + 3[G(x) - a_0]x^1 - 4G(x)x^2 = 0$$

$$[G(x) - 3 - (-2)x] + 3[G(x) - 3]x - 4G(x)x^2 = 0$$

$$[G(x) - 3 + 2x] + 3[G(x) - 3]x - 4x^2 G(x) = 0$$

$$G(x) - 3 + 2x + 3x\,G(x) - 9x - 4x^2 G(x) = 0$$

$$G(x) + 3x\,G(x) - 4x^2 G(x) - 3 + 2x - 9x = 0$$

$$G(x)[1 + 3x - 4x^2] - 3 - 7x = 0$$

$$G(x)[1 + 3x - 4x^2] = 3 + 7x$$

$$G(x) = \frac{3+7x}{1+3x-4x^2}$$

$$\begin{array}{c} -4 \\ 4 \;|\; -1 \\ 1 \end{array}$$

$$G(x) = \frac{3+7x}{(1-x)(1+4x)}$$

Ga

Now Consider, $\frac{3+7x}{(1-x)(1+4x)} = \frac{A}{1-x} + \frac{B}{1+4x} \longrightarrow (1)$

$$= \frac{A(1+4x)+B(1-x)}{(1-x)(1+4x)}$$

Compare, Nr on both side,

$$3+7x = A(1+4x)+B(1-x) \longrightarrow (2)$$

Put $\boxed{x=1}$ in Equ (2)

$3+7(1) = A(1+4)+B(0)$

$10 = A(5)$

$5A = 10$

$A = 10/5$

$\boxed{A=2}$

Put $\boxed{x=-1/4}$ in Equ (2)

$3+7(-1/4) = A(0)+B[1-(-1/4)]$

$3-7/4 = B(1+1/4)$

$\frac{12-7}{4} = B(5/4)$

$\cancel{5/4} = B(\cancel{5/4})$

$1 = B \Rightarrow \boxed{B=1}$

Put the value of $\boxed{A=2}$ & $\boxed{B=1}$ in Equ (1)

$$(1) \Rightarrow \frac{3+7x}{(1-x)(1+4x)} = \frac{A}{1-x} + \frac{B}{1+4x}$$

$$G(x) = \frac{2}{1-x} + \frac{1}{1+4x}$$

Taking general terms for this generating function

$$a_n = 2(1)^n + (-4)^n //$$

(2) Solve the recurrence relation $a_n - 2a_{n-1} - 3a_{n-2} = 0$, $n \geq 2$ with initial value $a_0 = 3$, $a_1 = 1$ using generating function.

Soln:

Given that $a_n - 2a_{n-1} - 3a_{n-2} = 0 \mid n \geq 2$, $a_0 = 3$, $a_1 = 1$

Let $G(x) = \sum_{n=2}^{\infty} a_n x^n$ is generating function

multiply by $\sum_{n=2}^{\infty} x^n$ on both side

$$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 3\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$[G(x) - a_0 - a_1 x]x^0 - 2[G(x) - a_0]x - 3[G(x)]x^2 = 0$$

$$[G(x) - 3 - (1)x](1) - 2[G(x) - 3]x - 3x^2[G(x)] = 0$$

$$G(x) - 3 - x - 2x[G(x)] + 6x - 3x^2[G(x)] = 0$$

$$G(x) - 2x[G(x)] - 3x^2[G(x)] - 3 - x + 6x = 0$$

$$G(x) - 2x[G(x)] - 3x^2[G(x)] = 3 - 5x$$

$$G(x)[1 - 2x - 3x^2] = 3 - 5x$$

$$G(x) = \frac{3 - 5x}{1 - 2x - 3x^2}$$

$$\begin{array}{c} -3 \\ 1 \;\big|\; -3 \\ -2 \end{array}$$

$$G(x) = \frac{3 - 5x}{(1 + x)(1 - 3x)}$$

Now, Consider,

$$\frac{3 - 5x}{(1 + x)(1 - 3x)} = \frac{A}{1 + x} + \frac{B}{1 - 3x} \longrightarrow (1)$$

$$= \frac{A(1 - 3x) + B(1 + x)}{(1 + x)(1 - 3x)}$$

Compare Nr on both side.

$$3-5x = A(1-3x) + B(1+x) \longrightarrow (2)$$

Put $\boxed{x=-1}$ in Equ (2)

$$3-5(-1) = A[1-3(-1)] + 0$$

$$3+5 = A[1+3]$$

$$8 = 4A$$

$$4A = 8$$

$$A = 8/4$$

$$\boxed{A=2}$$

Put $\boxed{x=1/3}$ in Equ (2)

$$3-5/3 = 0 + B[1+1/3]$$

$$\frac{9-5}{3} = B\left[\frac{3+1}{3}\right]$$

$$\cancel{4/3} = B[\cancel{4/3}]$$

$$1 = B$$

$$\boxed{B=1}$$

Put the values of $\boxed{A=2}$ & $\boxed{B=1}$ in Eqn (1)

$$(1) \Rightarrow \frac{3-5x}{(1+x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-3x}$$

$$G(x) = \frac{2}{1+x} + \frac{1}{1-3x}$$

Taking general term for this generating function

$$a_n = 2(-1)^n + (3)^n$$

(3) Using generating function, solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$, $n \geq 2$ & $a_0 = 2$, $a_1 = 8$.

Soln

Let $G(x) = \sum_{n=2}^{\infty} a_n x^n$ be generating function

Given that, $a_n = 4a_{n-1} - 4a_{n-2} + 4^n$ / $n \geq 2$, $\boxed{a_0=2}$, $\boxed{a_1=8}$

ie, $a_n - 4a_{n-1} + 4a_{n-2} = 4^n$

Multiply by $\sum_{n=2}^{\infty} x^n$ on both sides

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$$\sum_{n=2}^{\infty} a_n x^n - 4\sum_{n=2}^{\infty} a_{n-1} x^n + 4\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} (4)^n x^n$$

$$[G(x) - a_0 - a_1 x] - 4[G(x) - a_0]x + 4[G(x)]x^2 = \frac{1}{1-4x} - 1 - 4x$$

$$[G(x) - 2 - 8x] - 4[G(x) - 2]x + 4x^2[G(x)] = \frac{1}{1-4x} - 1 - 4x$$

$$G(x) - 2 - \cancel{8x} - 4x[G(x)] + \cancel{8x} + 4x^2[G(x)] = \frac{1}{1-4x} - 1 - 4x$$

$$G(x) - 4x[G(x)] + 4x^2[G(x)] - 2 = \frac{1}{1-4x} - 1 - 4x$$

$$G(x) - 4x[G(x)] + 4x^2[G(x)] = \frac{1}{1-4x} + 2 - 1 - 4x$$

$$G(x)[1 - 4x + 4x^2] = \frac{1}{1-4x} + 1 - 4x$$

$$= \frac{1 + (1-4x)(1-4x)}{1-4x}$$

$$= \frac{1 + 1 - 4x - 4x + 16x^2}{1-4x}$$

$$G(x)[1 - 4x + 4x^2] = \frac{2 - 8x + 16x^2}{1-4x}$$

$$G(x) = \frac{2 - 8x + 16x^2}{(1-4x)(1-4x+4x^2)}$$

$$\begin{array}{c} 4 \\ -2 \mid -2 \\ \hline -4 \end{array}$$

$$G(x) = \frac{2 - 8x + 16x^2}{(1-4x)(1-2x)(1-2x)}$$

$$G(x) = \frac{2 - 8x + 16x^2}{(1-4x)(1-2x)^2}$$

Now Consider,

$$\frac{2-8x+16x^2}{(1-4x)(1-2x)^2} = \frac{A}{1-4x} + \frac{B}{(1-2x)} + \frac{C}{(1-2x)^2} \longrightarrow (1)$$

$$= \frac{A(1-2x)^2 + B(1-2x)(1-4x) + C(1-4x)}{(1-4x)(1-2x)^2}$$

Compare Nr on both sides

$$2-8x+16x^2 = A(1-2x)^2 + B(1-2x)(1-4x) + C(1-4x) \longrightarrow (2)$$

Put $\boxed{x=1/4}$ in Eqn (2)

$$2 - \cancel{8}/4 + \cancel{16}(1/\cancel{16}) = A(1 - 2/4)^2$$

$$\cancel{2} - \cancel{2} + 1 = A(1-1/2)^2$$

$$1 = A(1/2)^2$$

$$1 = \frac{A}{4}$$

$$\boxed{A=4}$$

Put $\boxed{x=1/2}$ in Eqn (2)

$$2 - 8/2 + \frac{16}{4} = \cancel{B(1-4/2)(1-4/}$$

$$= C[1 - 4/2]$$

$$2 - \cancel{4} + \cancel{4} = C[1-2]$$

$$2 = C(-1)$$

$$2 = -C$$

$$\boxed{C=-2}$$

Put $\boxed{x=0}$ in Eqn (2)

$$2 = A(1)^2 + B(1) + C(1)$$

$$2 = A + B + C$$

$$2 = 4 + B - 2$$

$$2 = 2 + B \Rightarrow 2+B = 2 \Rightarrow B = 2-2 \Rightarrow \boxed{B=0}$$

Put the values of $\boxed{A=4}$ $\boxed{B=0}$, & $\boxed{C=-2}$ in Eqn (1)

$$(1) \Rightarrow \frac{2-8x+16x^2}{(1-4x)(1-2x)^2} = \frac{4}{1-4x} + 0 - \frac{2}{(1-2x)^2}$$

$$= \frac{4}{1-4x} - \frac{2}{(1-2x)^2}$$

$$G(x) = \frac{4}{1-4x} - \frac{2}{(1-2x)^2}$$

Taking general term for this generating function

$$a_n = 4(4)^n - 2(n+1)(2)^n$$

(4) use generating functions to solve the recurrence relation $a_n = 3a_{n-1} + 1$; $n \geq 1$ with $a_0 = 1$

Sol^n: Let $G(x) = \sum_{n=1}^{\infty} a_n x^n$ is the generating function.

Given that $a_n = 3a_{n-1} + 1$ / $n \geq 1$, $\boxed{a_0 = 1}$

$$a_n - 3a_{n-1} = 1$$

multiply by $\sum_{n=1}^{\infty} x^n$ on both sides

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} (1) x^n$$

$$[G(x) - a_0]x^0 - 3[G(x)]x = x + x^2 + x^3 + \cdots$$

$$[G(x) - 1](1) - 3x[G(x)] = x[1 + x + x^2 + \cdots]$$

$$G(x) - 3x[G(x)] - 1 = x\left[\frac{1}{1-x}\right] \quad \left\{ \text{W.K.T } (1-x)^{-1} = 1 + x + x^2 + \cdots \right\}$$

$$G(x) - 3x[G(x)] = 1 + x\left[\frac{1}{1-x}\right]$$

$$G(x)[1-3x] = 1 + \frac{x}{1-x}$$

$$= \frac{1 \cancel{-x} \cancel{+x}}{1-x}$$

$$G(x)[1-3x] = \frac{1}{1-x}$$

$$G(x) = \frac{1}{(1-x)(1-3x)}$$

Now Consider, $\dfrac{1}{(1-x)(1-3x)} = \dfrac{A}{(1-x)} + \dfrac{B}{(1-3x)} \longrightarrow (1)$

$$= \frac{A(1-3x) + B(1-x)}{(1-x)(1-3x)}$$

Compare N_r on both side.

$$1 = A(1-3x) + B(1-x) \longrightarrow (2)$$

Put $\boxed{x=1}$ in Eqn (2)

$$1 = A[1-3(1)] + 0$$
$$= A[1-3]$$
$$1 = A(-2)$$
$$-2A = 1$$
$$\boxed{A = -1/2}$$

Put $\boxed{x = 1/3}$

$$1 = 0 + B[1-1/3]$$
$$= B\left[\frac{3-1}{3}\right]$$
$$1 = B(2/3)$$
$$2/3\, B = 1 \Rightarrow \boxed{B = 3/2}$$

Put the values of $\boxed{A = -1/2}$ & $\boxed{B = 3/2}$ in Eqn (1)

$$(1) \Rightarrow \frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$= \frac{-1/2}{(1-x)} + \frac{3/2}{1-3x}$$

$$G(x) = (3/2)\frac{1}{(1-3x)} - (1/2)\frac{1}{(1-x)}$$

Taking general term for this generating function

$$a_n = (3/2)(3)^n - 1/2(1)^n, \quad n \geq 0$$

(5) Use generating function to solve the recurrence relation $a_{n+2}+3a_{n+1}+2a_n=3^n$, $n\geq 0$, with $a_0=1, a_1=2$.

Sol/n Let $\sum_{n=0}^{\infty} a^n x^n$ is the generation function.

Given that $a_{n+2}+3a_{n+1}+2a_n=3^n$ / $n\geq 0$, $\boxed{a_0=1}$ & $\boxed{a_1=2}$

multiply by $\sum_{n=0}^{\infty} x^n$ on both sides.

$$\sum_{n=0}^{\infty} a_{n+2}x^n+3\sum_{n=0}^{\infty} a_{n+1}x^n+2\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty}(3)^n x^n$$

$$[G(x)-a_0-a_1x]x^{-2}+3[G(x)-a_0]x^{-1}+2[G(x)]x^0=\frac{1}{1-3x}$$

put the value of $\boxed{a_0=1}$ & $\boxed{a_1=2}$

$$[G(x)-(1-2x)]\frac{1}{x^2}+3[G(x)-1]\frac{1}{x}+2[G(x)]=\frac{1}{1-3x}$$

multiply of x^2 on both side

$$[G(x)-1-2x]\cancel{\frac{x^2}{x^2}}+3[G(x)-1]\cancel{\frac{x^2}{x}}+2[G(x)]x^2=\frac{x^2}{1-3x}$$

$$[G(x)-1-2x]+3[G(x)-1]x+2x^2[G(x)]=\frac{x^2}{1-3x}$$

$$G(x)-1-2x+3x[G(x)]-3x+2x^2[G(x)]=\frac{x^2}{1-3x}$$

$$G(x)+3x[G(x)]+2x^2[G(x)]-1-5x=\frac{x^2}{1-3x}$$

$$G(x)+3x[G(x)]+2x^2[G(x)]=\frac{x^2}{1-3x}+1+5x$$

$$G(x)\left[1+3x+2x^2\right] = \frac{x^2}{1-3x} + 1 + 5x$$

$$= \frac{x^2 + (1+5x)(1-3x)}{(1-3x)}$$

$$= \frac{x^2 + 1 - 3x + 5x - 15x^2}{(1-3x)}$$

$$= \frac{x^2 + 1 + 2x - 15x^2}{(1-3x)}$$

$$G(x)\left[1+3x+2x^2\right] = \frac{1+2x-14x^2}{(1-3x)}$$

$$G(x) = \frac{1+2x-14x^2}{(1-3x)(1+3x+2x^2)} \qquad \begin{array}{c} 2 \\ 1 \mid 2 \\ 3 \end{array}$$

$$G(x) = \frac{1+2x-14x^2}{(1-3x)(1+x)(1+2x)}$$

Now Consider,

$$\frac{1+2x-14x^2}{(1-3x)(1+x)(1+2x)} = \frac{A}{1-3x} + \frac{B}{1+x} + \frac{C}{1+2x} \longrightarrow (1)$$

$$= \frac{A(1+x)(1+2x) + B(1-3x)(1+2x) + C(1-3x)(1+x)}{(1-3x)(1+x)(1+2x)}$$

Compare Nr on both side,

$$1+2x-14x^2 = A(1+x)(1+2x) + B(1-3x)(1+2x) + C(1-3x)(1+x) \longrightarrow (2)$$

Put $\boxed{x = 1/3}$ in Eq (2)

$$1 + 2/3 - 14(1/9) = A(1+1/3)(1+2/3)$$

$$1 + 2/3 - \frac{14}{9} = A\left(\frac{3+1}{3}\right)\left(\frac{3+2}{3}\right)$$

$$\frac{9+6-14}{9} = A(4/3)(5/3)$$

$$1/9 = \frac{20}{9}A$$

$$1 = 20A \implies \boxed{A = 1/20}$$

Put $\boxed{x=-1}$ in Eqn ②

$$1-2-14 = B(1+3)(1-2)$$
$$1-16 = B(4)(-1)$$
$$\cancel{+}15 = \cancel{+}4B$$
$$4B = 15$$
$$\boxed{B = \frac{15}{4}}$$

Put $\boxed{x=-1/2}$ in Eqn ②

$$1+\cancel{2}(-1/\cancel{2})-14(-1/2)^2 = C[1-3(-1/2)][1-1/2]$$
$$\cancel{1}-\cancel{1}-14(1/4) = C[1+3/2][1/2]$$
$$-14/4 = C\left[\frac{2+3}{2}\right][1/2]$$
$$-14/4 = C\left[\frac{5}{2}\right][1/2]$$
$$-14/\cancel{4} = C\left(\frac{5}{\cancel{4}}\right)$$
$$5C = -14$$
$$\boxed{C = -14/5}$$

Put the values of $\boxed{A = 1/20}$ $\boxed{B = \frac{15}{4}}$ & $\boxed{C = -14/5}$ in Eqn ①

$$\frac{1+2x-14x^2}{(1-3x)(1+x)(1+2x)} = \frac{A}{(1-3x)} + \frac{B}{(1+x)} + \frac{C}{1+2x}$$

$$= \frac{(1/20)}{(1-3x)} + \frac{(15/4)}{(1+x)} + \frac{(-14/5)}{(1+2x)}$$

$$G(x) = (1/20)\frac{1}{1-3x} + (15/4)\frac{1}{1+x} - (14/5)\frac{1}{1+2x}$$

Taking general term for this generating function

$$a_n = (1/20)(3)^n + \left(\frac{15}{4}\right)(-1)^n - \left(\frac{14}{5}\right)(-2)^n$$

(6) Solve the recurrence relation $y_{n+2} - 4y_{n+1} + 3y_n = 0$ with $y_0 = 2$; $y_1 = 4$ using the generating function.

Soln: Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ is the generating function.

Given that $y_{n+2} - 4y_{n+1} + 3y_n = 0$ / $n \geq 0$, $y_0 = 2$; $y_1 = 4$

ie, $a_{n+2} - 4a_{n+1} + 3a_n = 0$; $\boxed{a_0 = 2}$ & $\boxed{a_1 = 4}$

multiply by $\sum_{n=0}^{\infty} x^n$ on both sides.

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 4\sum_{n=0}^{\infty} a_{n+1} x^n + 3\sum_{n=0} a_n x^n = 0$$

$$[G(x) - a_0 - a_1 x]x^{-2} - 4[G(x) - a_0]x^{-1} + 3[G(x)]x^0 = 0$$

multiply by x^2 on both sides

$$[G(x) - a_0 - a_1 x]x^0 - 4[G(x) - a_0]x + 3[G(x)]x^2 = 0$$

Put $\boxed{a_0 = 2}$ & $\boxed{a_1 = 4}$

$$[G(x) - 2 - 4x] - 4[G(x) - 2]x + 3x^2[G(x)] = 0$$

$$G(x) - 2 - 4x - 4x G(x) + 8x + 3x^2[G(x)] = 0$$

$$G(x) - 4x[G(x)] + 3x^2[G(x)] - 2 + 4x = 0$$

$$G(x) - 4x[G(x)] + 3x^2[G(x)] = 2 - 4x$$

$$G(x)[1 - 4x + 3x^2] = 2 - 4x$$

$$G(x)\left[1-4x+3x^2\right] = 2-4x$$

$$G(x) = \frac{2-4x}{1-4x+3x^2}$$

$$\begin{array}{c|c} & 3 \\ -1 & -3 \\ \hline & -4 \end{array}$$

$$G(x) = \frac{2-4x}{(1-x)(1-3x)}$$

Now Consider,

$$\frac{2-4x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x} \longrightarrow (1)$$

$$= \frac{A(1-3x)+B(1-x)}{(1-x)(1-3x)}$$

Compare Nr on both sides

$$2-4x = A(1-3x)+B(1-x) \longrightarrow (2)$$

Put $\boxed{x=1}$ in Eqn (2)

$$2-4(1) = A(1-3)$$

$$2-4 = A(-2)$$

$$-\cancel{2} = -\cancel{2}A$$

$$1 = A$$

$$\boxed{A=1}$$

Put $\boxed{x=1/3}$ in Eqn (2)

$$2-4/3 = B(1-1/3)$$

$$\frac{6-4}{3} = B\left(\frac{3-1}{3}\right)$$

$$\cancel{2/3} = B(\cancel{2/3})$$

$$1 = B \Rightarrow \boxed{B=1}$$

Put the value of $\boxed{A=1}$ & $\boxed{B=1}$ in Eq (1)

$$\frac{2-4x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$G(x) = \frac{1}{1-x} + \frac{1}{1-3x}$$

Taking general terms for this generating function

$$a_n = (1)^n + (3)^n, \quad n \geq 0.$$

Inclusin - Exclusion Principal

① $|A \cup B| = |A| + |B| - |A \cap B|$

② $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$

① 40 Computer Programmers interviewed for a job. 25 knew JAVA, 28 knew ORACLE, and 7 knew neither language. How many knew both languages?

Sol/ Given that, Total no - of Programmers = 40.

JAVA known ~~Programmers~~ languages = $|A| = 25$

ORACLE known ~~Programmers~~ languages = $|B| = 28$

& 7 of them known neither language.

$|A \cup B| = 40 - 7 = 33$

Computer Programmers who knew both languages are.

$$|A \cap B| = |A| + |B| - |A \cup B|$$
$$= 25 + 28 - 33$$
$$= 53 - 33$$
$$= 20$$

(2) In a survey of 300 students, 64 had taken a maths course, 94 had taken a English course, 58 had taken a Computer Course, 28 had taken both maths and Computer course, 26 had taken both English and maths course, 22 had taken both a English and computer course. 14 had taken all three course. How many students were surveyed who had taken non of the three courses.?

Soln:

Let A be the no. of maths course students

B be the no. of English course students

C be the no. of Computer course students

ie, $|A| = 64$; $|B| = 94$; $|C| = 58$; $|A \cap B| = 28$; $|B \cap C| = 26$

$|A \cap C| = 22$; $|A \cap B \cap C| = 14$.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 64 + 94 + 58 - 28 - 26 - 22 + 14$$

$$= 154$$

The total no. of students $= 300$

$\therefore$ students who had taken none of the courses

$$= 300 - 154$$

$$= 146$$

(3) Among the first 1000 Positive integers: Determine the integers which are not divisible by 5, nor 7, nor by 9.

Soln: Let A be the number of integers divisible by 5

B be the number of integers divisible by 7

C be the number of integers divisible by 9

ie ~~$|A| = 100$~~ The total no-of. students $= 1000$

$$|A| = \left|\frac{1000}{5}\right| = 200; \quad |B| = \left|\frac{1000}{7}\right| = 142; \quad |C| = \left|\frac{1000}{9}\right| = 111$$

$$|A \cap B| = \left|\frac{1000}{LCM(5,7)}\right| = \left|\frac{1000}{5 \times 7}\right| = 28$$

$$|B \cap C| = \left|\frac{1000}{LCM(7,9)}\right| = \left|\frac{1000}{7 \times 9}\right| = 22$$

$$|A \cap C| = \left|\frac{1000}{LCM(5,9)}\right| = \left|\frac{1000}{5 \times 9}\right| = 15$$

$$|A \cap B \cap| = \left|\frac{1000}{LCM(5,7,9)}\right| = \left|\frac{1000}{5 \times 7 \times 9}\right| = 3$$

The no-of-integers divisible by 5, 7 & 9.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$\cancel{= 2001} = 200 + 142 + 111 - 28 - 22 - 15 + 3$$

$$= 391.$$

The no-of. integers not divisible by 5, nor by 7, nor by 9.

$=$ Total. no-of. students $-$ integers divisible by 5, 7, 9

$= 1000 - 391$

$= 609$ //

(4) In a class of 50 students, 20 students play foot ball and 16 students play hockey. It is found that 10 students play both the games. Find the no. of. students who play neither.

Soln: Let the total no. of. students = 50

A be the no. of. students playing foot ball

B be the no. of. students playing hockey.

$|A| = 20$; $|B| = 16$; $|A \cap B| = 10$.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 20 + 16 - 10$$

$$= 26.$$

The no. of. students play neither game.

= total no. of. students − no. of. students play atleast one game

$= 50 - 26$

$= 24$ //

(5) In a survey of 100 students, it was found that 40 studied maths, 64 studied physics, 35 studied chemistry, 1 studied all the three subject, 25 studied maths and physics, 3 studied maths and chemistry, 20 studied physics and chemistry. Use the inclusion and exclusion, find the no. of. students who studied

Chemistry only, and the no. of students who studied none of these subjects.

Soln

Let A be the no. of students studied maths

B be the no. of students studied Physics

C be the no. of students studied chemistry.

$|A| = 40; \ |B| = 64; \ |C| = 35; \ |A \cap B| = 25; \ |B \cap C| = 20$

$|A \cap C| = 3; \ |A \cap B \cap C| = 1.$

i) The no. of students studied atleast one subject

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 40 + 64 + 35 - 25 - 3 - 20 + 1$$

$$\boxed{|A \cup B \cup C| = 92.}$$

The total no. of students = 100

ii) The no. of students studied none of the subjects

= Total no. of students − no. of students studied atleast one subject

$= 100 - 92$

$= 8$

(or)

$$|\overline{A \cup B \cup C}| = n - |A \cup B \cup C|$$

$$= 100 - 92$$

$$= 8$$

(iii) The no. of students studied chemistry subject only

$= |C| - |A \cap C| - |B \cap C| - |A \cap B \cap C|$

$= 35 - 3 - 20 - 1$

$= 35 - 24$

$= 11$

(5) Find the number of integers between 1 and 250 that are not divisible by any of the integer 2, 3, 5 and 7.

Soln

Let A be set of integer between 1 to 250 divisible by 2

B " " " " by 3

C " " " " by 5

D " " " " by 7

$|A| = \left[\frac{250}{2}\right] = 125$; $|B| = \left[\frac{250}{3}\right] = 83$; $|C| = \left[\frac{250}{5}\right] = 50$

$|D| = \left[\frac{250}{7}\right] = 35$

$|A \cap B| = \left[\frac{250}{2 \times 3}\right] = \frac{250}{6} = 41$; $|A \cap C| = \left[\frac{250}{2 \times 5}\right] = \frac{250}{10} = 25$

$|A \cap D| = \left[\frac{250}{2 \times 7}\right] = \frac{250}{14} = 17$; $|B \cap C| = \left[\frac{250}{3 \times 5}\right] = \frac{250}{15} = 16$

$|B \cap D| = \left[\frac{250}{3 \times 7}\right] = \frac{250}{21} = 11$; $|C \cap D| = \left[\frac{250}{5 \times 7}\right] = \frac{250}{35} = 7$

$|A \cap B \cap C| = \left[\frac{250}{2 \times 3 \times 5}\right] = \frac{250}{30} = 8$

$|A \cap B \cap D| = \left[\frac{250}{2 \times 3 \times 7}\right] = \frac{250}{42} = 5$

$|A \cap C \cap D| = \left[\frac{250}{2 \times 5 \times 7}\right] = \frac{250}{70} = 3$

$$|B\cap C\cap D| = \left[\frac{250}{3\times5\times7}\right] = \frac{250}{105} = 2.$$

$$|A\cap B\cap C\cap D| = \frac{250}{2\times3\times5\times7} = \frac{250}{210} = 1$$

$$|A\cup B\cup C\cup D| = |A| + |B| + |C| + |D| - |A\cap B| - |A\cap C| - |A\cap D| - |B\cap C|$$
$$- |B\cap D| - |C\cap D| + |A\cap B\cap C| + |A\cap B\cap D|$$
$$+ |A\cap C\cap D| + |B\cap C\cap D| - |A\cap B\cap C\cap D|$$
$$= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$
$$= 293 - 117 + 18 - 1$$
$$= 311 - 118$$
$$\boxed{|A\cup B\cup C\cup D| = 193}$$

The integers between 1 to 250 not divisible by 2,3,5,7

is $|\overline{A\cup B\cup C\cup D}| = n - |A\cup B\cup C\cup D|$

$$= 250 - 193$$
$$= 57$$

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2024–2025

MA3354- DISCRETE MATHEMATICS

UNIT-3

GRAPHS

Prepare by

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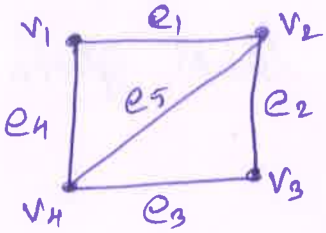
Department of Mathematics

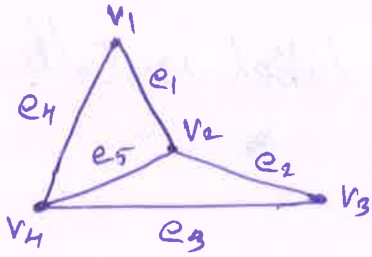
UNIT - 3 [GRAPHS]

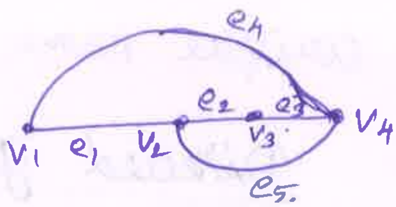
Chapter - 3.1 [Graphs and graph models]

Defn: **Graph** is a ordered pair of vertices and Edges and which is denoted by $G[V, E]$.

Example: Let $G = [V(G), E(G)]$, where $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$

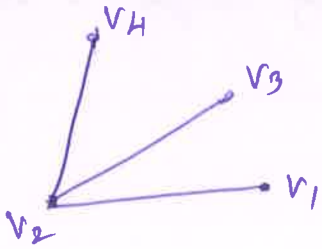






Defn: **Adjacent vertices**

Any pair of vertices which are connected by an Edge in a graph is called adjacent vertices.

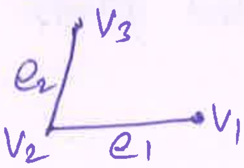


Here, v_1, v_2 ; v_2, v_4 ; v_2, v_3 are adjacent vertices.

v_1, v_3 ; v_3, v_4 ; v_1, v_4 are not adjacent

Defn: **Adjacent edges**

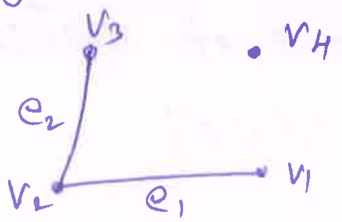
If two distinct Edges are incident with a common vertex then they are called adjacent Edges.



Here e_1 & e_2 are incident with a common vertex v_2.

Defn: <u>Isolated vertex</u>

In any graph, a vertex which is not adjacent to any other vertex is called an isolated vertex.



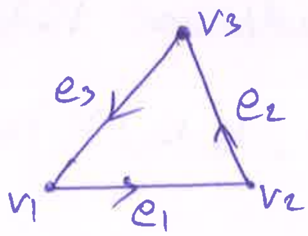
Defn: <u>Label graph</u>

A graph in which each vertex is assigned a unique name or label is called a label graph.

Defn: <u>Directed graph</u>

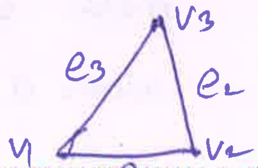
In a graph $G(V,E)$ an edge which is associated with an ordered pair of vertices is called a directed graph.

ie, A graph in which every edge is directed is called a directed graph



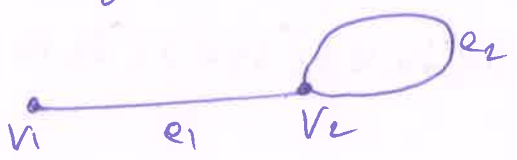
Defn: <u>undirected graph</u>

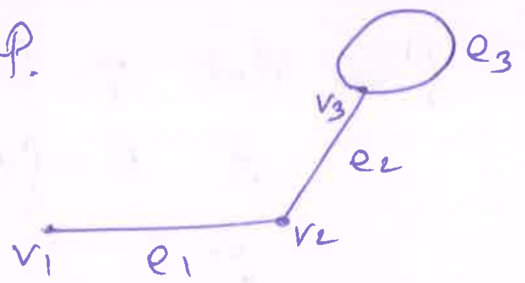
A graph in which every edge is undirected is called a undirected graph.



Defn: Loop

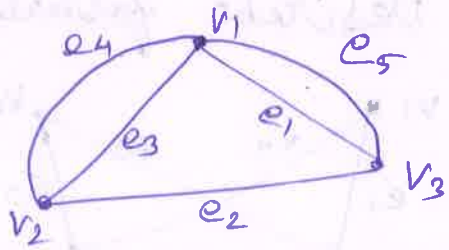
A loop is an edge whose vertices are equal i.e., An edge of a graph which joins a vertex to itself is called a loop.





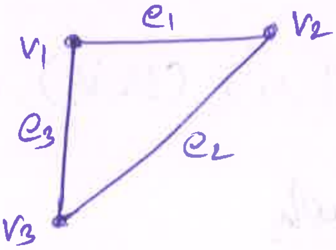
Defn: Parallel Edges (or) multiple Edges

multiple Edges are edges having the same pair of vertices.



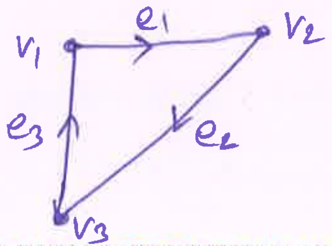
Defn: Simple graph

A simple graph is a graph having no loops (or) multiple Edges.



Defn: Simple directed graph

When a directed graph has no loops and has no multiple directed edges, it is called simple directed graph.



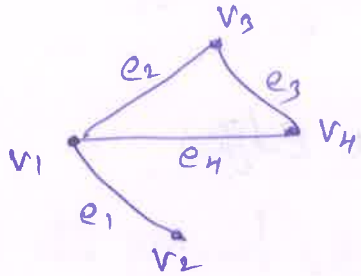
Example: Draw a diagram for the following graph.

$G = G(V, E)$, where $V = \{v_1, v_2, v_3, v_4\}$

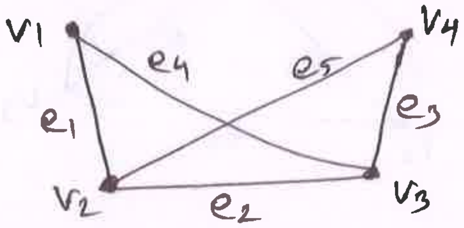
and $E = \{(v_1, v_2), (v_4, v_1), (v_3, v_1), (v_3, v_4)\}$

Soln: Given that $V = \{v_1, v_2, v_3, v_4\}$

& $E = \{(v_1, v_2), (v_4, v_1), (v_3, v_1), (v_3, v_4)\}$



Example: Describe formally the graph given below.



Soln: $G = (V, E)$

$V = \{v_1, v_2, v_3, v_4\}$ & $E = \{e_1, e_2, e_3, e_4, e_5\}$

$\therefore E(G) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_1, v_3), (v_2, v_4)\}$

Defn: Pseudograph

Graph that may include loops, and possible multiple edges connecting the same pair of vertices are sometimes called pseudographs.

Chapter - 3.2 [Graph Terminology and Special Types of Graph]

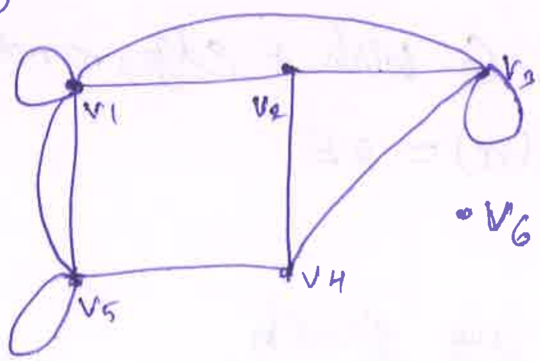
Defn: Two vertices u and v in an undirected graph G are called adjacent (or neighbour) in G. if u, v are endpoints of an edge of G.

i.e. The vertices u and v are called endpoints of an edge associated with (u, v).

Defn: The degree of a vertex

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. It is denoted by deg( ).

Eg:



$deg(v_1) = 6$

$deg(v_2) = 3$

$deg(v_3) = 5$

$deg(v_4) = 3$

$deg(v_5) = 5$

$deg(v_6) = 0$

Note:

1) Let G be an undirected graph with $|E|$ Edges and $|V| = n$ vertices, then $\sum_{i=1}^{n} \deg(V_i) = 2|E|$.

2) In any graph, the no. of. vertices of odd degree is Even.

3) A vertex of degree one is called a Pendant or End vertex in G.

Example: How many Edges are there in a graph with 10 vertices each of degree six?

Soln: Sum of the degree of the 10 vertices is $(6)(10) = 60$.

$\therefore 2e = 60 \Rightarrow e = 60/2 \Rightarrow \boxed{e = 30}$

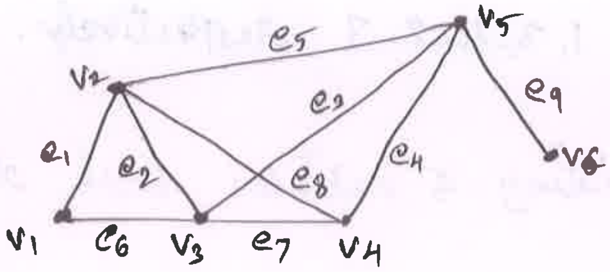
The Handshaking Theorem:-

Statement: For any graph G with E Edges and V vertices $V_1, V_2, V_3 \ldots V_n$, $\sum_{i=1}^{n} d(V_i) = 2E$

Proof: Let $G = G(V, E)$ be any graph, where $V = \{V_1, V_2, V_3 \ldots V_n\}$ and $E = \{e_1, e_2, e_3 \ldots e_n\}$

Since, each edges contributes twice as a degree, the sum of the degree of all vertices in G is twice as the no. of. Edges in G. i.e., $\sum_{i=1}^{n} d(V_i) = 2|E|$
$= 2e$.

Example - ① verify the handshaking theorem for the graph



Solu:

To Prove $\sum_{i=1}^{n} d(v_i) = 2e$

$= 2(\text{no. of. Edges})$

ie $\sum_{i=1}^{6} d(V_i) = 2(9) = 18.$

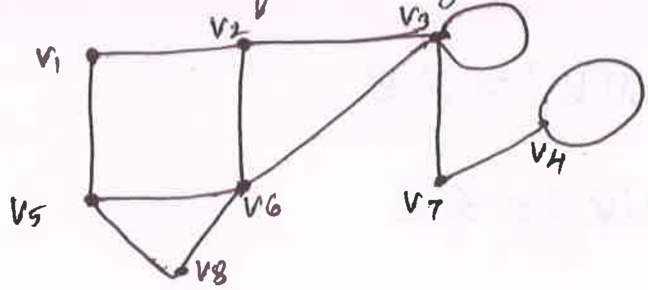
Now, $\sum_{i=1}^{6} d(V_i) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6)$

$= 2 + 4 + 4 + 3 + 4 + 1$

$= 18.$

Hence the theorem is true.

Example - ② verify that the sum of the degree of all the vertices is even for the graph.



Solu: The sum of degree of all the vertices is

$= d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) + d(v_7) + d(v_8)$

$= 2 + 2 + 5 + 3 + 3 + 4 + 2 + 2$

$= 24$ which is Even.

Example-③ Show that there does not exist a graph with 5 vertices with degrees 1, 3, 4, 2, 3 respectively.

Sol/n: Given graph consisting 5 vertices with degree 1, 3, 4, 2, 3.

Since, W.K.T $\sum d(v_i) = 2e$ (Even)

Here, $v_1 + v_2 + v_3 + v_4 + v_5 = 1+3+4+2+3$

$= 13$ (not Even)

There is no graph with 5 vertices with degree 1, 3, 4, 2, 3.

Example-④ An undirected graph G has 16 edges and all the vertices are of degree 2. Find the number of vertices.

Sol/n: Given that G be the graph with edges $e = 16$

and $d(v_i) = 2$, $i = 1, 2, 3, \ldots n$

Since, $\sum d(v_i) = 2e$

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\sum_{i=1}^{n} 2 = 2(16)$$

$$2n = 32$$

$$n = 32/2 \Rightarrow \boxed{n = 16}$$

no. of vertices $= 16$.

Example - (5)

Let G be a graph with 10 vertices. If 4 vertices has degree 4 and 6 vertices has degree 5, then find the number of Edges of G?

Soln:

Let G be a graph with vertices V_i, $i = 1,2,3,\ldots 10$

Let $d(V_i) = 4$, $i = 1,2,3,4$

$d(V_j) = 5$, $j = 5,6,7,8,9,10$

Since, $\sum_{i=1}^{4} d(V_i) + \sum_{j=5}^{10} d(V_j) = 2e$

$[V_1 + V_2 + V_3 + V_4] + [V_5 + V_6 + V_7 + V_8 + V_9 + V_{10}] = 2e$

$[4+4+4+4] + [5+5+5+5+5+5] = 2e$

$16 + 30 = 2e$

$46 = 2e \Rightarrow 2e = 46 \Rightarrow e = 46/2$

$\boxed{e = 23}$

Example: (6)

Show that the maximum number of Edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Soln: Use handshaking theorem

i.e, $\sum_{i=1}^{n} d(V_i) = 2e$.

Where e is the no. of. Edges with n vertices in the graph G.

i.e, $d(V_1) + d(V_2) + \cdots + d(V_n) = 2e \longrightarrow (1)$

Since, we know that the maximum degree of Each vertex in the graph can be $(n-1)$

$$① \Rightarrow (n-1)+(n-1)+ \cdots + n \text{ terms} = 2e$$

$$\Rightarrow n(n-1) = 2e$$

$$\Rightarrow 2e = n(n-1)$$

$$e = \frac{n(n-1)}{2}$$

Hence, the maximum no-of-Edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Example-⑦

How many Edges are there in a graph with 10 vertices each of degree 5?

Soln:

Given that G has 10 vertices of Each of the degree 5.

Let $V_1, V_2, \ldots, V_{10}$ be the vertices of G.

then $d(V_1) = d(V_2) = \cdots = d(V_{10}) = 5$

$\therefore$ by hand-shaking theorem.

$$d(V_1) + d(V_2) + \cdots + d(V_{10}) = 2e$$

$$5+5+ \cdots +5 \text{ (10 times)} = 2e$$

$$10 \times 5 = 2e$$

$$50 = 2e$$

$$2e = 50 \Rightarrow e = 50/2$$

$$\boxed{e = 25}$$

Theorem:-

The maximum number of Edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Solu:-

Let G be a simple graph with 'n' vertices

To prove this theorem by mathematical induction method.

Given that $P(n) = \frac{n(n-1)}{2}$

Step-1: Put $\boxed{n=1}$

$$P(1) = \frac{1(1-1)}{2} = 0$$

If $n=1$, then G is a graph with one vertex and 0 Edges

The result is true for $n=1$.

Step-2

Assume that $P(k)$ is true

ie $P(k) = \frac{k(k-1)}{2}$

Step-3

To prove $P(k+1)$ is true

ie $P(k+1) = \frac{(k+1)(k+1-1)}{2}$

$$P(k+1) = \frac{(k+1)(k)}{2}$$

Which is true for $P(k+1)$

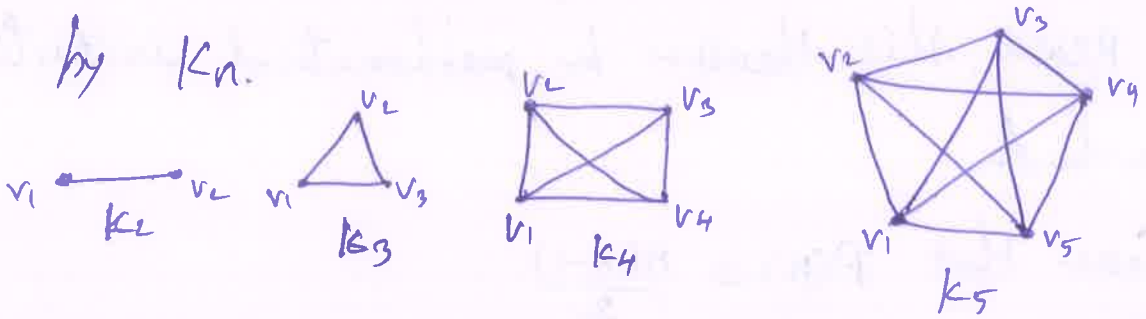
$\therefore$ $P(k)$ is true

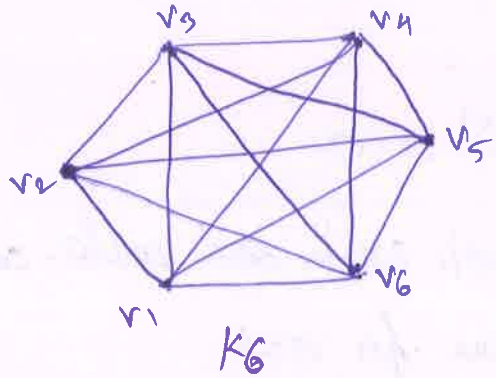
$\therefore$ G has maximum $\frac{n(n-1)}{2}$ Edges.

Defn. <u>Complete graph</u>

A simple graph G is said to be complete if every pair of vertices are adjacent.

A complete graph with n vertices is denoted by K_n.





Eg-① How many vertices and how many edges of K_n?

<u>Soln</u>

In K_n graph, n-vertices & $\frac{n(n-1)}{2}$ edges.

Eg-② What is the degree sequence of K_n, where n is a positive integer? Explain your answer.

<u>Soln</u>

Each of the n vertices is adjacent to each of the other $n-1$ vertices.

So the degree sequence is $n-1, n-1, n-1, \ldots$ (n entries)

Eg (3) Find the degree sequence of each of the following graph (i) K_4, (ii) K_5 (iii) K_6.

Soln

W.K.T. For K_n have the degree sequence as

$$(n-1), (n-1), \ldots, (n-1) \text{ [n terms]}$$

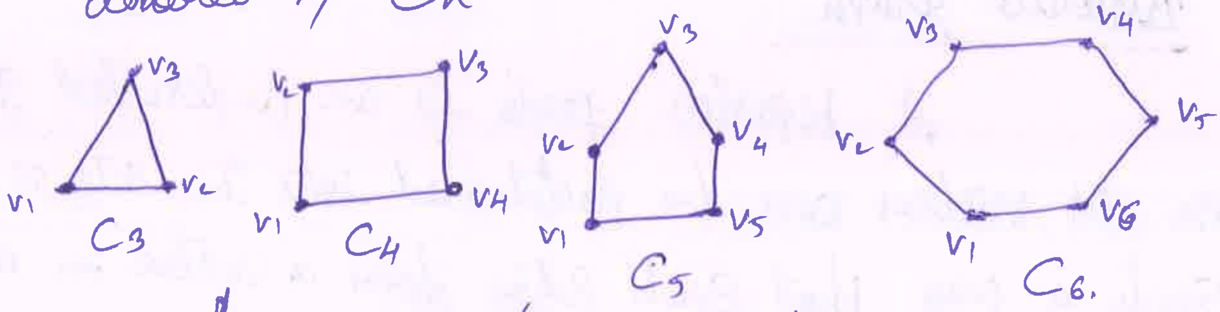
(i) K_4 = 3, 3, 3, 3

(ii) K_5 = 4, 4, 4, 4, 4

(iii) K_6 = 5, 5, 5, 5, 5

Defn: **Cycle Graph**

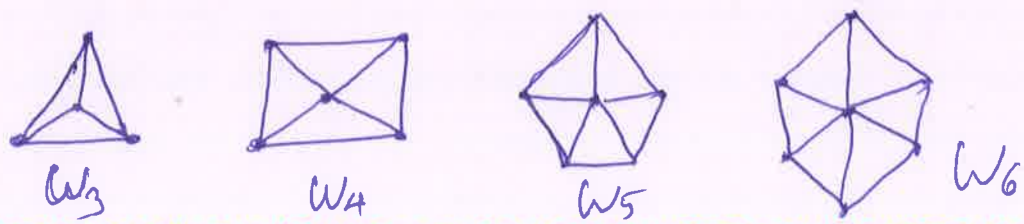
A cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n' and denoted by C_n



It has n-vertices & n edges.

Defn: **Wheel Graph**

A wheel graph of order n is obtained by joining a new vertex called "Hub" to each vertex of a cycle graph of order n-1, and denoted by W_n. [n+1 vertices & 2n Edges]



Defn: **Regular graph:**

A graph in which all vertices are of Equal degree is called a regular graph.

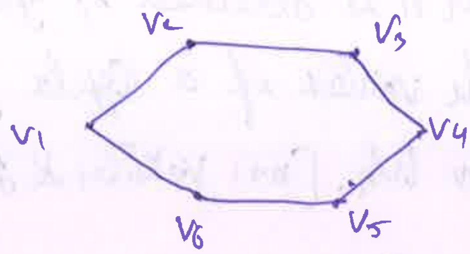
If the degree of Each vertex is 'r' then the graph is called a regular graph of degree r.

Note:
(i) Every null graph is regular of degree Zero
(ii) The Complete graph K_n of degree $n-1$
(iii) If G has n vertices and is regular of degree 'r' then G has $(\frac{1}{2})rn$ Edges (i.e.) $\frac{rn}{2}$ Edges
(iv) Every Complete graph is regular graph.

Defn: **Bipartite graph**

A bipartite graph is an undirected graph whose set vertices can be partitioned into two sets M & N in such a way that Each Edge join a vertex in M to a vertex in N and no Edge joins Either two vertices in M or two vertices in N.

Eg:

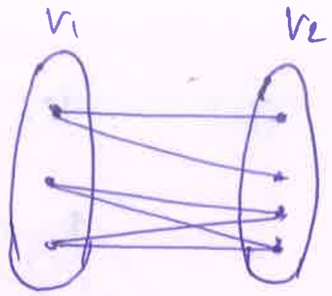


(OR)

Bipartite graph:-

Let G be a graph we say that G is bipartite graph if the vertex set of G can be partitioned into two sets V_1 and V_2 such that every edge of G has one end in V_1 and another end V_2

Eg:

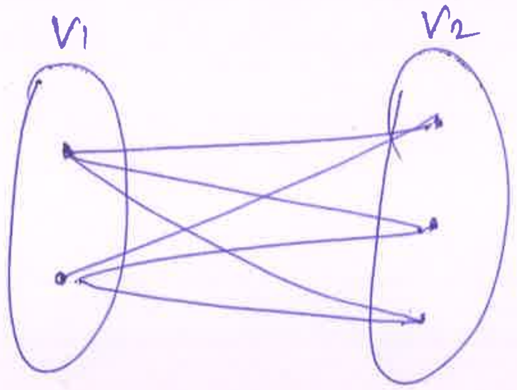


Defn: **Complete Bipartite Graph**

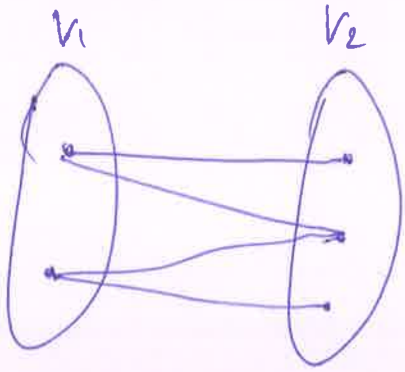
A bipartite graph G with bipartition V_1 and V_2 is called complete bipartite graph if every vertices of V_1 is adjacent to every vertex of V_2.

A complete bipartite graph is denoted by $K_{m,n}$

Eg:



Complete

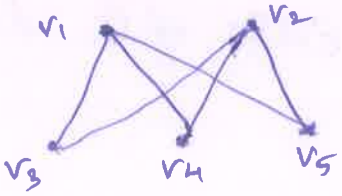


not Complete

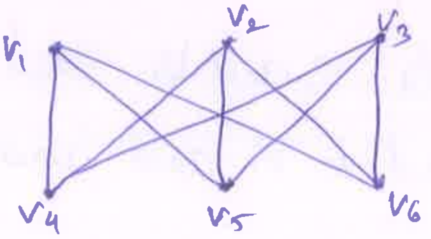
Eg: Draw the Complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{2,6}$ and $K_{3,5}$.

Sol:

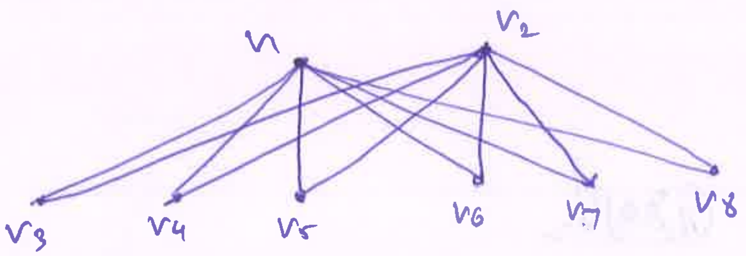
(i) $K_{2,3}$



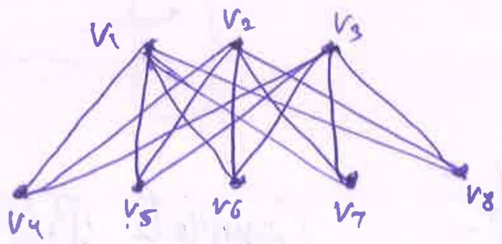
(ii) $K_{3,3}$



(iii) $K_{2,6}$



(iv) $K_{3,5}$



① In $K_{m,n}$ graphs have.
m+n vertices & mn Edges

② The degree sequence of the $K_{2,3}$ graph have, = 3, 3, 2, 2, 2

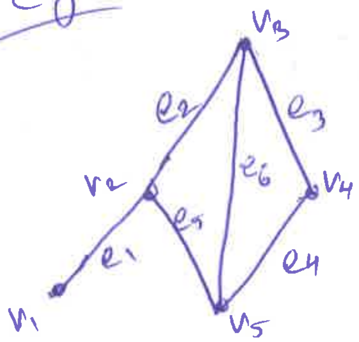
Defn: Subgraph

A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

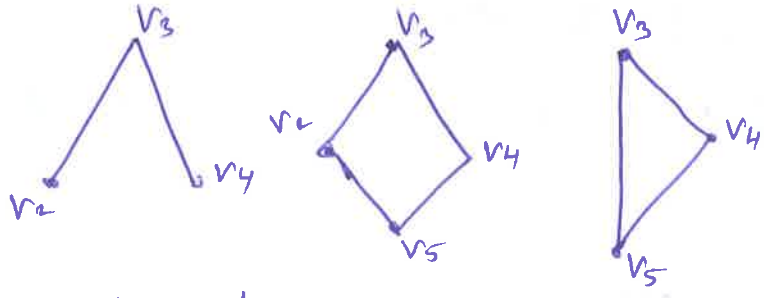
(OR)

A graph H is said to be a subgraph of G if,
(i) All the vertices of H are in G.
(ii) All the Edges of H are in G
(iii) Adjacency is Preserved in H Exactly as in G.

Eg:



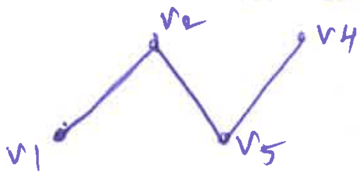
Graph



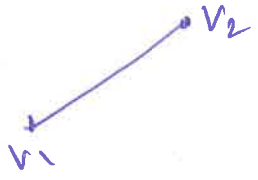
Subgraphs

Vertex deleting subgraphs

(i) By deleting v_3

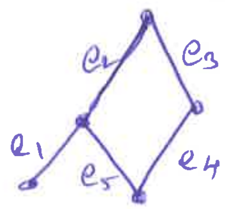


(ii) By deleting v_3, v_5

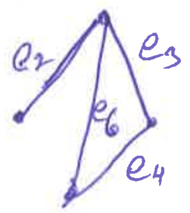


Edges deleting subgraphs

(i) By deleting e_6

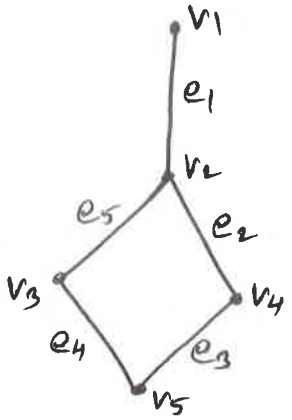


(ii) By deleting e_1, e_5



Problem: Define subgraph. Find all the subgraph of the following graph by deleting an edge.

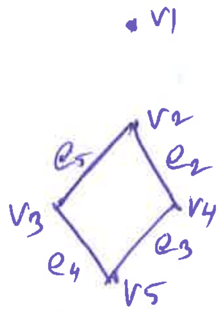
Sol: <u>Subgraph</u> – A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$, where $W \subseteq V$ and $F \subseteq E$.



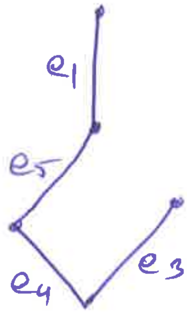
Graph

Subgraph are

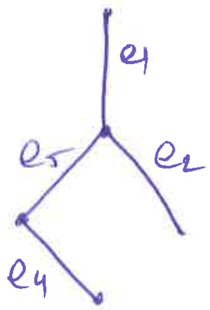
(i) H_1: By deleting e_1



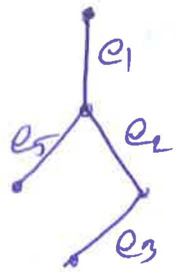
(ii) H_2: By deleting e_2



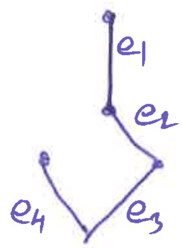
(iii) H_3: By deleting e_3



(iv) H_4: By deleting e_4



(v) H_5: By deleting e_5



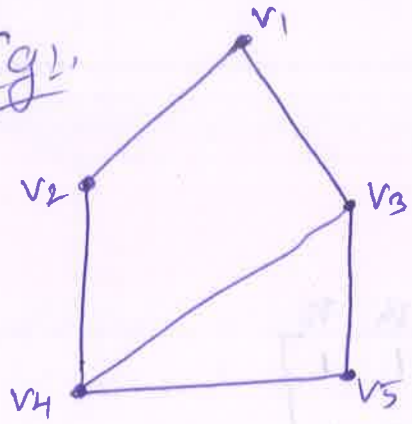
Chapter - 3.3

Representing Graphs and Graph Isomorphism

Defn: Matrix representation of graphs and Digraphs

We can represent a simple graph in the form of Edge list or in the form of adjacency lists which are may be useful in computer programming.

Eg:



Vertex	Adjacency vertices
v_1	$\longrightarrow v_2, v_3$
v_2	$\longrightarrow v_1, v_4$
v_3	$\longrightarrow v_1, v_4, v_5$
v_4	$\longrightarrow v_2, v_3, v_5$
v_5	$\longrightarrow v_3, v_4$

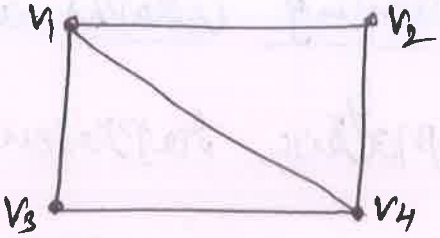
Defn: Adjacency matrix

Let G be a graph then the adjacency matrix of G is defined by $A = [a_{ij}] = \begin{cases} 1, & \text{if } v_i \text{ adjacency to } v_j \\ 0, & \text{otherwise} \end{cases}$

Eg:

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Eg: Find the adjacency list of vertices and the adjacency matrix.



Sol^n

(i) Vertex $\quad$ Adjacent vertices

$V_1 \longrightarrow V_2, V_3, V_4$

$V_2 \longrightarrow V_1, V_4$

$V_3 \longrightarrow V_1, V_4$

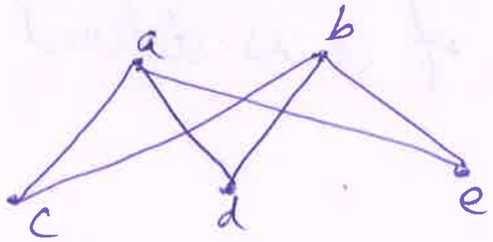
$V_4 \longrightarrow V_1, V_2, V_3$

(ii) Adjacency matrix

$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Eg: Write the adjacency matrix of $K_{2,3}$.

Sol^n $\quad$ $K_{2,3}$ graph is



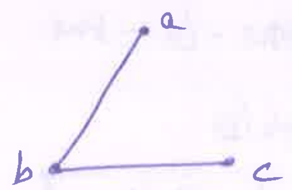
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Eg: Draw a graph of the given adjacency matrix.

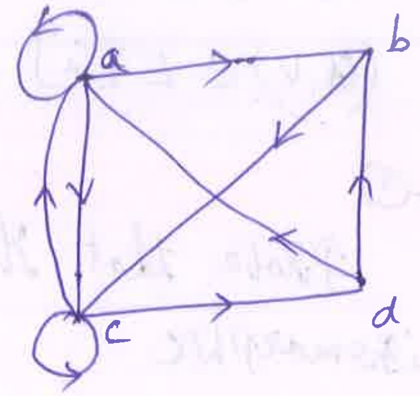
(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Soln:

(a) Let
$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



(b) Let
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



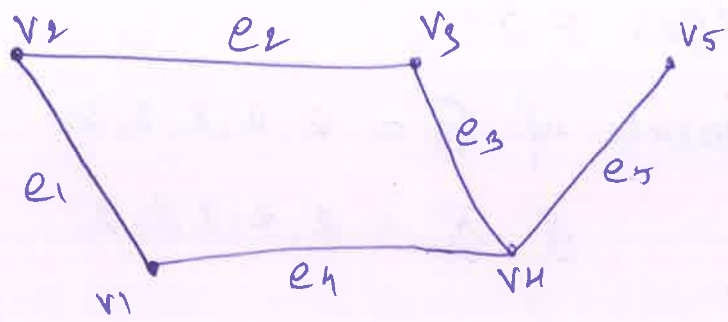
Defn: Incidence matrix

Let G be a graph with n vertices.

Let $V = \{v_1, v_2, v_3, \dots, v_n\}$ and $E = \{e_1, e_2, e_3, \dots, e_n\}$

Define $n \times m$ matrix $I_G = [m_{ij}]_{n \times m}$.

Where $m_{ij} = \begin{cases} 1, & \text{when } v_i \text{ is incident with } e_j \\ 0, & \end{cases}$



$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Defn: **Isomorphic Graphs**

The graphs G and G' are isomorphic if there is a function $f: V(G) \to V(G')$ from the vertices of G to the vertices of G' such that

(i) f is one-to-one

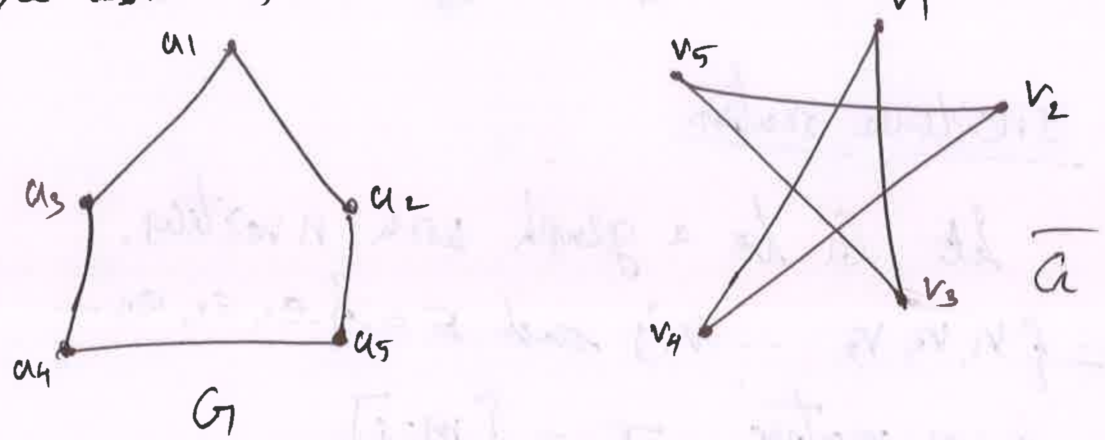
(ii) f is onto

(iii) For each pair of vertices u and v of G

$$(u,v) \in E[G] \iff [f(u), f(v)] \in E(G')$$

Problem - 1

Prove that the graph G and $\bar{G}$ given below are isomorphic.



Soln:

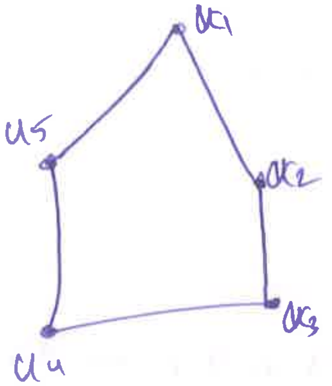
Both the Graphs G and $\bar{G}$ have

(i) Same no. of vertices $= 5$

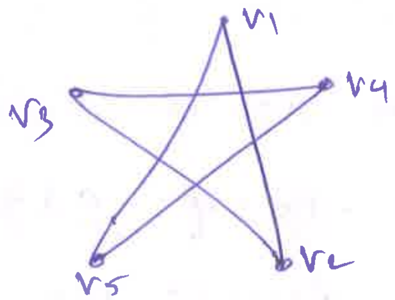
(ii) Same no. of Edges $= 5$

(iii) The degree sequence of $G = 2,2,2,2,2$

& $\bar{G} = 2,2,2,2,2$



Re-numbering the vertices of H



The adjacent matrix of G

$$A(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacent matrix of $\bar{G}$

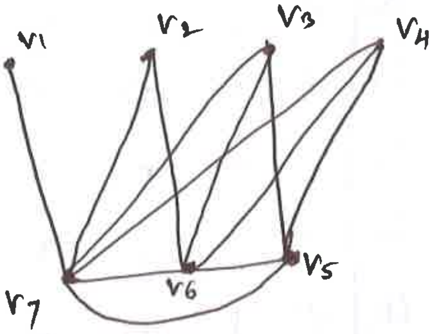
$$A(\bar{G}) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\therefore \; A(G) = A(\bar{G})$$

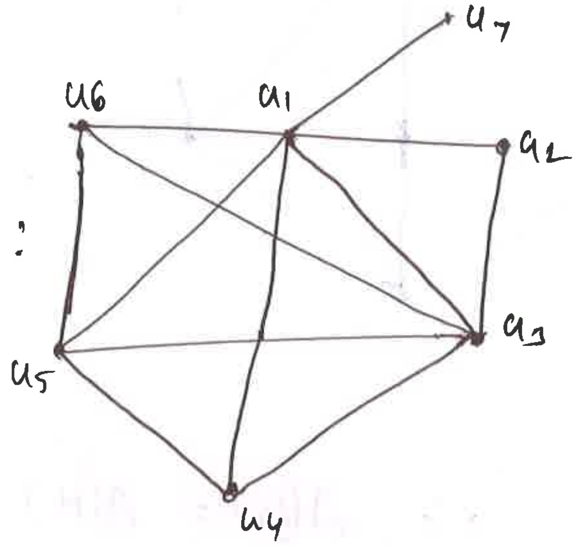
$\therefore$ G & $\bar{G}$ are isomorphic

Problem-② Show that the following Graphs are isomorphic

G:



H:



Soln:

Both the graphs G & H have

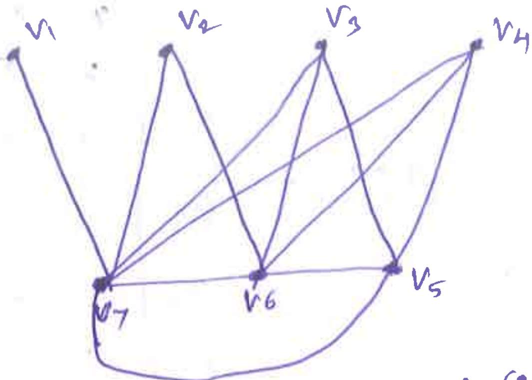
i) Same no. of vertices = 7

ii) Same no. of edges = 12

iii) ~~Same~~ degree squence G: 1, 2, 3, 3, 4, 5, 6 = 24

H: 6, 2, 5, 3, 4, 3, 1 = 24

G:

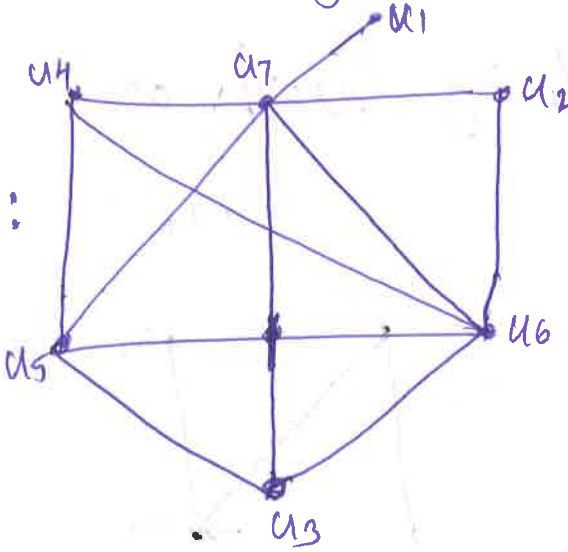


The adjacency matrix of G

$$A(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Renumbering the vertices of H

H:



The adjacency matrix of H

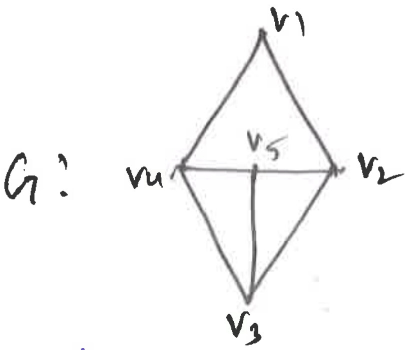
$$A(H) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore$ $A(G) = A(H)$

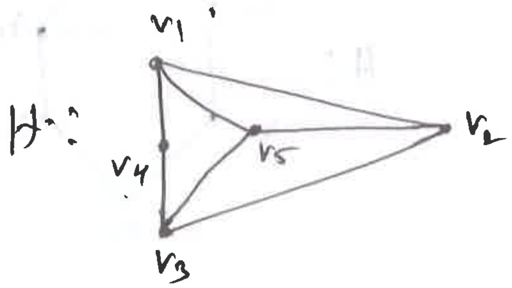
$\therefore$ G & H are isomorphic

Problem-(3) Establish the isomorphism of the following pairs of graphs, by considering their adjacency matrix.

G:



H:



Soln

Both the Graphs G and H have

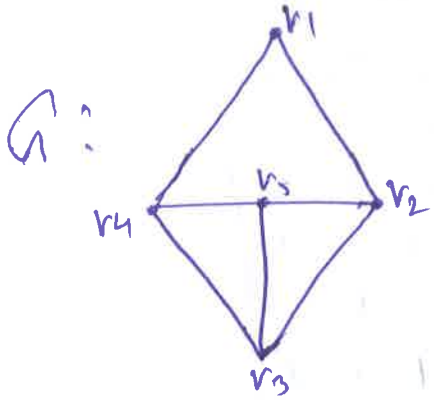
(i) Same no. of. vertices $= 5$

(ii) Same no. of. Edges $= 5$

(iii) The degree sequence of $G = 2, 3, 3, 3, 3 = 14$

& $H = 3, 3, 3, 2, 3 = 14$

G:

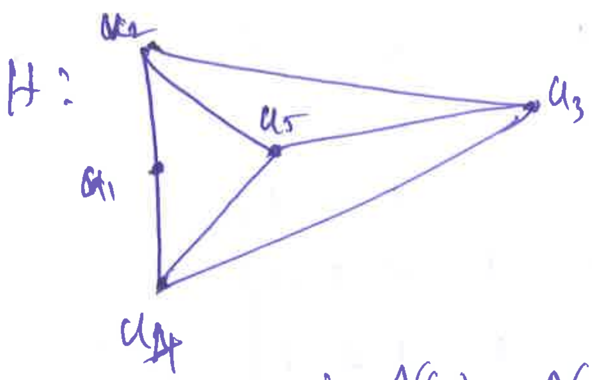


The adjacency matrix of G

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Renumbering the vertices of H

H:



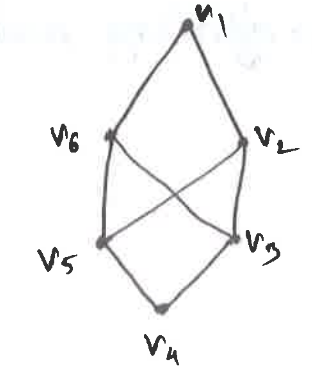
$$A(H) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore A(G) = A(H)$

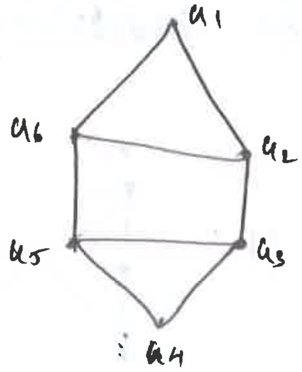
$\therefore G$ & H are isomorphic.

Problem - ④ Check whether the graphs are isomorphic or not?

G:



H:



Sol^n:

Both the graphs G & H have

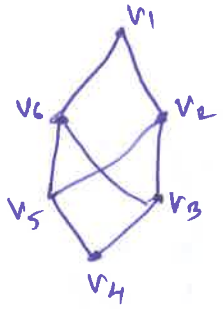
i) Same no. of vertices = 6

ii) Same no. of edges = 8

iii) The degree sequence G: 2, 3, 3, 2, 3, 3 = 16

H: 2, 3, 3, 2, 3, 3 = 16

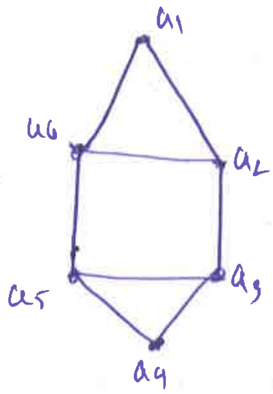
G:



The adjacency matrix of G.

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

H:



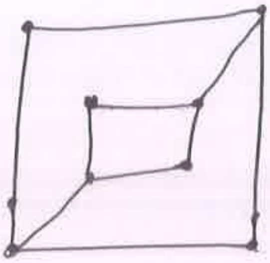
$$A(H) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$A(G) = A(H)$

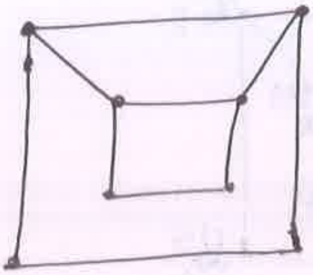
$\therefore$ G & H are isomorphic.

Problem-(5) Check whether the graphs are isomorphic or not?

G_1:



G_2:



Soln: Both the graphs G_1 & G_2 have

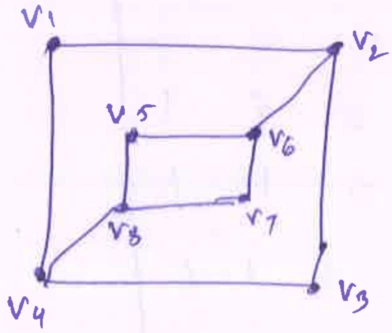
i) Same no. of vertices $= 8$

ii) Same no. of Edges $= 10$

iii) The degree sequence G_1: $2, 3, 2, 3, 2, 3, 2, 3 = 20$

G_2: $3, 3, 2, 2, 3, 3, 2, 2 = 20$

G_1:

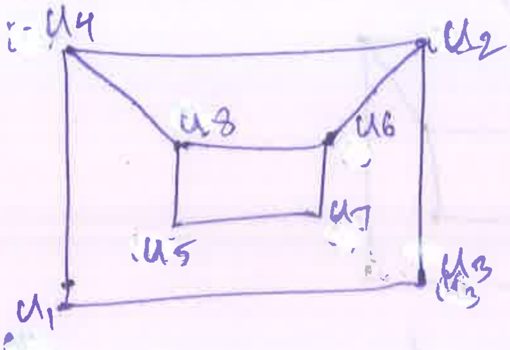


The adjacency matrix of G_1

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Renumbering the vertices of G_2

G_2 :



The adjacency matrix of G_2

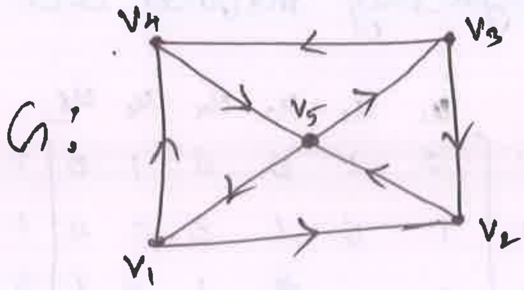
$$A(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

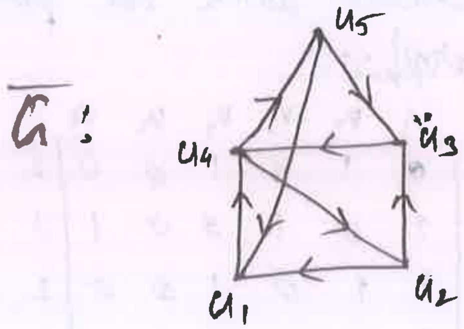
$A(G_1) \neq A(G_2)$

$\therefore$ G_1 & G_2 are not isomorphic.

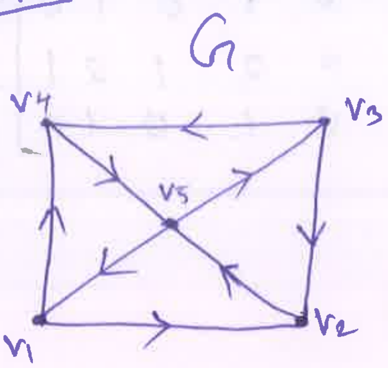
Problem - (6)

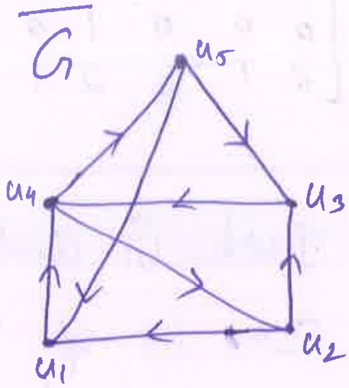
Show that the Digraphs are isomorphic





Soln:





Both the graphs G and $\bar{G}$ have

(i) Same no. of Vertices $= 5$

(ii) Same no. of Edges $= 8$

Consider the indegree and outdegree of vertices G & $\bar{G}$

G	deg + in degree	deg − out degree
v_1	1	2
v_2	2	1
v_3	1	2
v_4	2	1
v_5	2	2

$\bar{G}$	deg + in degree	deg − out degree
u_1	2	1
u_2	1	2
u_3	2	1
u_4	2	2
u_5	1	2

Now, $f(v_1) = u_5$, $f(v_2) = u_1$, $f(v_3) = u_2$, $f(v_4) = u_3$ & $f(v_5) = u_4$

Clearly f is one to one and onto

$\therefore$ G and $\bar{G}$ are isomorphic.

Problem-(7) Examine whether the following two graphs G & G' associated with the following adjacency matrices are isomorphic.

$$G: \begin{array}{c|cccccc|c} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & \\ v_1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ v_2 & 1 & 0 & 1 & 0 & 0 & 1 & 3 \\ v_3 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\ v_4 & 1 & 0 & 1 & 0 & 1 & 0 & 3 \\ v_5 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \\ v_6 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ \end{array} \quad \underline{14}$$

$$G': \begin{array}{c|cccccc|c} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & \\ u_1 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ u_2 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\ u_3 & 0 & 1 & 0 & 1 & 0 & 1 & 3 \\ u_4 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ u_5 & 1 & 0 & 0 & 1 & 0 & 1 & 3 \\ u_6 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ \end{array} \quad \underline{14}$$

Soln) Both G and G' have

(i) Same no. of vertices (6)

(ii) Same no. of Edges (7)

(iii) The degree of sequence

G : 2, 3, 2, 3, 2, 2

H : 2, 2, 3, 2, 3, 2

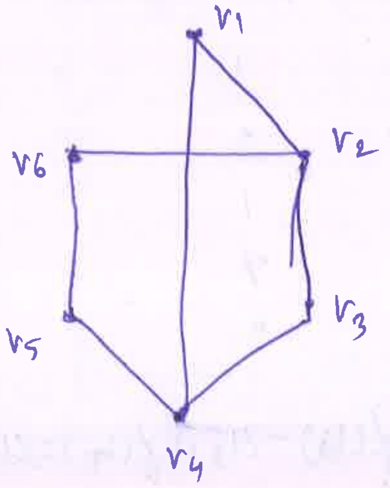
W.K.T $\sum d(v_i) = 2e$

$14 = 2e$

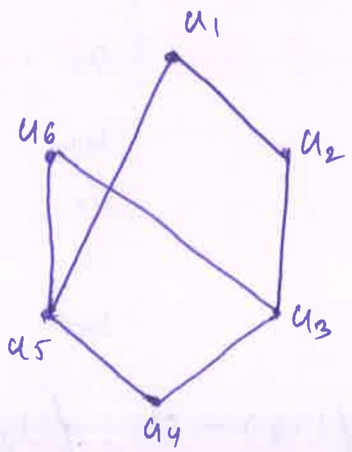
$14/2 = e$

$\boxed{7 = e}$

G



G'



Since, G and G' is isomorphic.

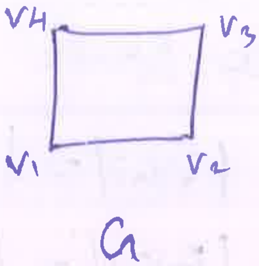
Defn: <u>Complement of a graph</u>

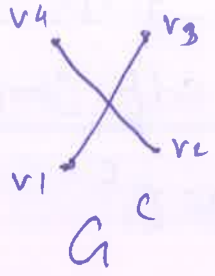
Let G be a simple graph we say that G^c (or) G' is a complement of G if

(i) the vertices of G^c are the vertices of G.

ii) two vertices are joined in G iff they are not adjacent in G'.

Eg:





$\boxed{G \cup G^c = K_n}$

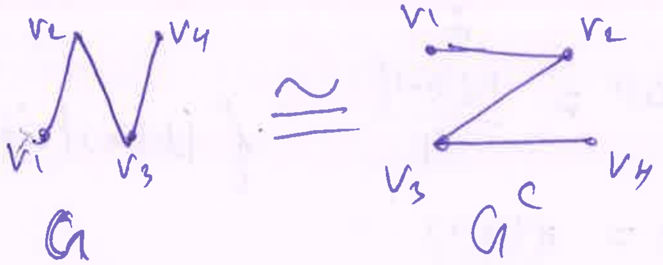
Defn: <u>Self Complement graph</u>

Let G be a simple graph we say that G is self complement if G is isomorphic to G^c.

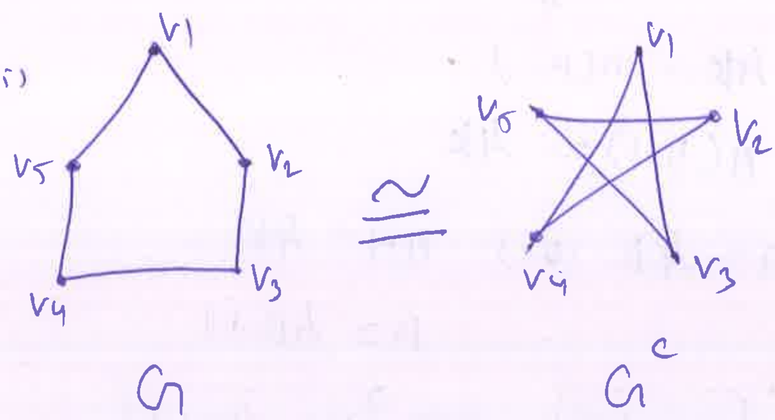
i.e. $G \cong G^c$.

Eg:

(i)



(ii)



Problem - ⑪ Show that self complementary with n vertices then $n \equiv 0$ (or) (mod 4)

(or)

Every self complementary graph has $4k$ (or) $4k+1$ vertices.

Soln:

Let G be graph with 'n' vertices

Since, G is self complementary graph

$$G \cong G^c \qquad \therefore \; |V(G)| = |V(G^c)|$$

$$|E(G)| = |E(G^c)|$$

W.K.T

$$G \cup G^c = K_n \text{ (max edges in } G)$$

$$|E(G)| + |E(G^c)| = \frac{n(n-1)}{2}$$

$$|E(G)| + |E(G)| = \frac{n(n-1)}{2}$$

$$2|E(G)| = \frac{n(n-1)}{2}$$

$$|E(G)| = \frac{n(n-1)}{4} \qquad \{ |E(G)| \text{ is integer (say } k)$$

$$k = \frac{n(n-1)}{4}$$

$$4k = n(n-1)$$

ie, $n(n-1) = 4k$

$\Rightarrow n = 4k$ (or) $n-1 = 4k$

$$n = 4k+1$$

$\therefore \boxed{n = 4k}$ (or) $\boxed{n = 4k+1}$

Problem-② Using incident matrix of G. Show that the sum of the degree of vertices of G is equal to twice of the no. of Edges.

Soln: Let G be a graph with 'm' vertices & 'n' Edges

The incident matrix is defined by

$$B = \begin{cases} 1, & \text{if } e_i \text{ is incidence with } v_i \\ 0, & \text{otherwise} \end{cases}$$

The vertices of G be $\{v_1, v_2, v_3 --- v_n\}$

The edges of G be $\{e_1, e_2, e_3 --- e_n\}$

$$B = \begin{array}{c|c|c} & e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_n & \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & & & \\ 1 & 0 & 0 & \cdots \end{bmatrix} & \begin{matrix} d(v_1) \\ d(v_2) \\ d(v_3) \\ \vdots \\ d(v_m) \end{matrix} \\ & 2 \quad 2 \quad 2 \quad \cdots \quad 2 & \end{array}$$

Since, Each edge is incident with exactly 2 vertices

$\therefore$ Each column sum is $= 2$.

W.K.T sum of 1^{st} row $= d(v_1)$

sum of 2^{nd} row $= d(v_2)$

$\vdots$

sum of m^{th} row $= d(v_m)$

Since, In any matrix, row sum = column sum

$d(v_1) + d(v_2) + \cdots + d(v_m) = 2 + 2 + 2 \cdots + 2$

$\sum_{i=1}^{m} d(v_i) = 2n$, where n is no. of Edges

Chapter - 3.4 [Connectivity]

Defn: **Path**

A Path in a multigraph G consist of an alternating sequence of vertices and Edges of the form

$V_0, e_1, V_1, e_2, V_2, e_3 \ldots\ldots e_{n-1}, V_{n-1}, e_n, V_n$

where Each Edge e_i contains the vertices V_{i-1} and V_i

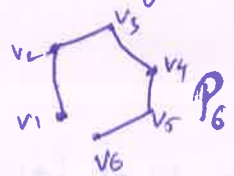
The number n of Edges is called the length of the Path.

Defn: **Circuit**

A Path of length ≥ 1 with no repeated Edges and whose End vertices are same is called circuit.

Defn: **Path graph**

A Path graph of order 'n' is obtained by removing an Edge from a C_n graph, denoted by P_n.



Defn: **Trail**

A trail from v to w is a Path from v to w that does not contain a repeated Edge.

Thus a trial from v to w is a Path of the form

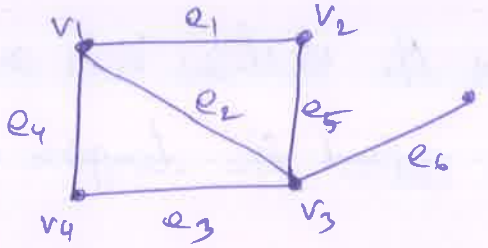
$V = v_0, e_1, v_1, e_2, \ldots\ldots v_{k-1}, e_k, v_k = w.$

where all the e_i are distinct.

Defn: Connected and disconnected graph

A graph G is a connected graph is there is atleast one path between every pair of vertices in G. otherwise G is disconnected graph.

Eg:



Theorem:

The maximum number of Edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.

Proof:

Let the no. of. vertices in the i^{th} component of G be n_i $(n_i > 1)$

then $n_1 + n_2 + n_3 + \cdots + n_k = n$ (or) $\sum_{i=1}^{k} n_i = n$ $\rightarrow (1)$

Hence,
$$\sum_{i=1}^{k} (n_i - 1) = n - k \longrightarrow (2)$$

$$\therefore \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$
$$= n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i,j} (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$
$\rightarrow (3)$

$$\Rightarrow \sum_{i=1}^{k} (n_i-1)^2 \le n^2+k^2-2nk \quad \{\because (3) \ge 0, \text{ as Each } n_i \ge 1\}$$

$$\Rightarrow \sum_{i=1}^{k} (n_i^2+1-2n_i) \le n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} (1-2n_i) \le n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 + (k-2n) \le n^2+k^2-2nk \quad \{\text{Using Eq } (2)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2+k^2-2nk-k+2n \longrightarrow (4)$$

Now the maximum no. of Edges in the i^{th} Component of

$$G = \tfrac{1}{2}\, n_i(n_i-1)$$

$\therefore$ The maximum no. of Edges of G

$$= \tfrac{1}{2}\sum_{i=1}^{k} n_i(n_i-1)$$

$$= \tfrac{1}{2}\sum_{i=1}^{k} (n_i^2-n_i)$$

$$= \tfrac{1}{2}\sum_{i=1}^{k} n_i^2 - \tfrac{1}{2}\sum_{i=1}^{k} n_i$$

$$= \tfrac{1}{2}\sum_{i=1}^{k} n_i^2 - \tfrac{1}{2}n \quad \{\text{Using Eqn } (1) \text{ \& Using Eq } (4)$$

$$\le \tfrac{1}{2}[n^2+k^2-2nk-k+2n]-\tfrac{1}{2}n$$

$$\le \tfrac{1}{2}[n^2+k^2-2nk-k+2n-n]$$

$$\le \tfrac{1}{2}[n^2+k^2-2nk-k+n]$$

$$\le \tfrac{1}{2}[(n-k)^2+(n-k)]$$

$$\le \tfrac{1}{2}(n-k)(n-k+1) \;//$$

Example Show that a simple graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Sol^n Suppose that G is not connected.

Then it has a component of k vertices for some k, $1 \le k \le n-1$.

The most edges G could have is

$$C[k,2] + C[n-k,2] = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2}$$

$$= \frac{k(k-1) + (n-k)(n-k-1)}{2}$$

$$= \frac{k^2 - \cancel{k} + n^2 - nk - n - nk + k^2 \cancel{+k}}{2}$$

$$= \frac{2k^2 - 2nk + n^2 - n}{2}$$

$$= k^2 - nk + \frac{(n^2-n)}{2}$$

which is quadratic function of f is minimized at $\boxed{k = n/2}$ and maximum at $\boxed{k=1}$ (or) $\boxed{k=n-1}$

Hence, if G is not connected, then the no. of edges does not exceed the value at 1 and $n-1$, namely $> \frac{(n-1)(n-2)}{2}$ edges.

Eulerian Graph

A Graph which contains an Eulerian Circuit is called Eulerian Graph.

Eulerian Cycle (or) Eulerian Circuit

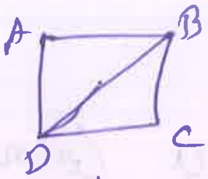
In Eulerian circuit satisfies

i) Starting and Ends Points are same

ii) Includes all the Edges Exactly once.

Eg:

Euler Graph



Not Euler Graph.

Note: Degree of Each vertex should be Even.

Hamiltonian Graph

A Graph which contains a Hamiltonian Cycle (or) Circuit is Called Hamiltonian Graph.

Hamiltonian Cycle (or) Hamiltonian Circuit

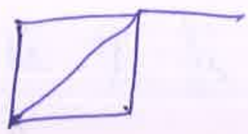
In Hamiltonian circuit satisfies

i) Starting and End Points are same

ii) Includes all the vertices exactly once.

Eg:

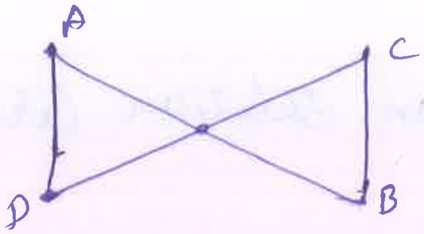
Hamilton Graph



Not Hamilton Graph

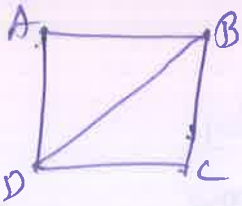
Eg:

(i)



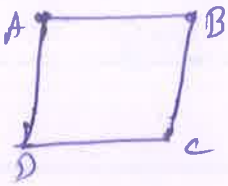
Euler Graph But not Hamilton Graph

(ii)



Hamilton Graph But not Euler Graph

(iii)



Euler Graph & Hamilton Graph

Theorem:

Show that a Connected Graph is Euler.

Iff all the vertices of G are of Even degree or no vertices of G are of odd degree.

(or)

State and Prove necessary and sufficient Condition for a Graph to be an Eulerian Graph.

Proof:

Suppose assume that G is Euler Graph

Then G has an Euler Cycle.

Say $C: \{u, v_1 v_2, v_3 - - \cdot v_n, u\}$

Clearly if v is an internal vertex

then $d(v) = 2 \times \left\{ \begin{array}{l} \text{The no. of. times } v \text{ occur in} \\ \text{the Euler circuit} \end{array} \right\}$ Even

Since, C starts and ends at u.

$$d(u) = 2 + 2 \times \begin{cases} \text{the no. of times } u \text{ occurs inside} \\ \text{the Euler cycle } C. \end{cases}$$

$\Rightarrow$ Every vertex of G has even degree.

Conversely,

Assume that, Every vertex of G has even degree

To prove that G is an Euler Graph.

Suppose, that G is not Euler Graph

ie, G has no Euler cycle

Since, Every vertex of G has even degree

G has a closed cycle.

Let 'C' be a closed cycle of maximum length

If C contains all the Edges of G, then C itself

a Euler cycle

$\therefore$ $E(G) - E(C)$ contains some other component G'

Since all the vertices of G are of even degree

and all the vertices of C are also even degree

$\therefore$ Every vertex of G' also contains even degree

G' contains a cycle C'

Since, G is Connected Graph

$\therefore$ C & C' should contain a common vertex 'w'

Now, join C and C' using w

$\Rightarrow$ $C \cup C'$ will be a new cycle in G

and $E(C \cup C') > E(C)$

$\Longrightarrow \Longleftarrow$ Since, $C \rightarrow$ maximum cycle

$\therefore$ G should be an Eulerian Graph.

ORE'S Theorem:-

If G is a simple graph with no. of vertices $(n \geq 3)$ and if $\deg(u) + \deg(v) \geq n \longrightarrow (1)$, for every pair of non-adjacent vertices u and v. then G is Hamiltonian.

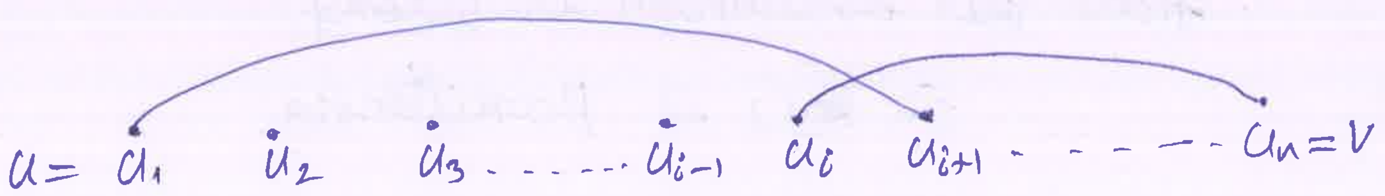
Proof:-

We shall prove the theorem by contradiction. We assume that there exists a non-Hamiltonian graph with n vertices satisfying the given condition (1) for every pair of non-adjacent vertices u and v.

Among all such non-Hamiltonian graphs, let G be a non-Hamiltonian graph with maximum no. of edges.

Because G is maximal non-Hamiltonian, it follows that there exist two non-adjacent vertices u and v in G such that addition of an edge joining u and v will result in a Hamiltonian graph.

Thus in G there is a Hamiltonian path $u = u_1, u_2, u_3, \ldots, u_n = v$ with u and v as the end vertices as show in figure



Define

$S = \{ i : u_i \text{ is adjacent to vertex } u \text{ in } G \}$

$T = \{ i : u_i \text{ is adjacent to vertex } v \text{ in } G. \}$

Clearly, $|S| = deg(u)$ & $|T| = deg(v)$ in G.

where $|X|$ denotes the no. of. Elements in a set X.

We assert that $S \cap T = \phi$ and $|S \cup T| \leq n-1$.

For if, $i \in S \cap T$, then the Edges (u, u_i) and (u_i, v) would be in G and then $u = u_1, u_2, u_3 \dots u_{i-1}, u_i, u_n, u_{n-1}, \dots u_{i+1}, u$. would form a Hamiltonian circuit in G,

which is a contradiction.

Father, $S \cup T \subset \{1, 2, 3, \dots n\}$. But since vertex $u_1 = u$ is neither adjacent to u nor adjacent to v.

$1 \notin S \cup T$.

Therefore, $|S \cup T| \leq n-1$

Now, we have $deg(u) + deg(v) = |S| + |T|$

$= |S \cup T|$, $S \cap T \neq 0$

$\leq n-1$

But this is a contradiction to our hypothesis

Hence our assumption is wrong.

So, G is Hamiltonian.

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MA3354-DISCRETE MATHEMATICS

UNIT-4

ALGEBRAIC STRUCTURES

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UNIT-4

ALGEBRAIC STRUCTURES

Algebraic systems – Semi groups and monoids - Groups – Subgroups – Homomorphism's – Normal subgroup and cosets – Lagrange's theorem – Definitions and examples of Rings and Fields.

UNIT-4

ALGEBRAIC STRUCTURES

Chapter-4.1

Defn: Group

A non Empty set G. together with a binary operation $*$ is said to form a group, if it satisfies the following Conditions.

i) Closure Property : $\forall a,b \in G$ s.t $a*b \in G$

ii) Associative Property : $\forall a,b,c \in G$ s.t $(a*b)*c = a*(b*c)$

iii) Identity Property : $\forall a,e \in G$ s.t $a*e = e*a = a$

iv) Inverse Property : $\forall a,a^{-1} \in G$ s.t $a*a^{-1} = a^{-1}*a = e$

Eg: $(Z,+)$ is a group

Defn: Abelian group

A group $(G,*)$ is said to be an abelian group if $a*b = b*a$, $\forall a,b \in G$

Eg: $(Z,+)$ is an abelian group

Defn: Semi group

A non Empty set S. together with a binary operation $*$ is called semi group if $*$ satisfies the following Condition:

(i) Closure Property: $\forall a,b \in G$ s.t $a*b \in G$

ii) Associative Property: $\forall a,b,c \in G$ s.t $(a*b)*c = a*(b*c)$

Eg: $(Z,\cdot)$ is a semigroup

Defn: **Monoid**

A non Empty set M, together with a binary operation $*$ is called a monoid if $*$ satisfies the following condition:

(i) Closure Property: $\forall a,b \in G$ s.t $a*b \in G$

(ii) Associative Property: $\forall a,b,c \in G$ s.t $(a*b)*c = a*(b*c)$

(iii) Identity Property: $\forall a,e \in G$ s.t $a*e = e*a = a$

Eg: $(Z,+)$ is a monoid.

Defn: **order of group**

Let G be a group under the binary operation $*$ The number of Elements in G is called the order of the group and is denoted by $o(G)$.

Eg: (i) let $G = \{1,-1,i,-i\}$ then $o(G) = 4$.

(ii) $o(Z) = \infty$

Defn: **Cyclic group**

A group G is called cyclic group, if there Exists an Element $a \in G$ s.t Each Element of G is Expressible as: $x = a^n = a.a.a \ldots a$ (n times).

where n is some integer.

the Element $a \in G$ is called a generator of G.

ie, $G = \langle a \rangle$ (or) $G = (a)$.

Eg:

(i) let $G = \{-1, 1, i, -i\}$ is a cyclic group, where $G = \langle i \rangle$

Notice that. $(i)^1 = i$; $(i)^2 = -1$, $(i)^3 = -i$; $(i)^4 = 1$; $(i)^5 = i$.

(ii) let $G = \{-1, 1\}$ is a cyclic group, where $G = \langle -1 \rangle$

Notice that $(-1)^1 = -1$; $(-1)^2 = 1$, $(-1)^3 = -1$, ...

Defn: **Subgroup**

A non-Empty subset H of a group G ($H \subseteq G$) is a subgroup of G iff $a, b \in H \Rightarrow ab^{-1} \in H$.

Eg: $(Z, +)$ is a subgroup of group $(R, +)$

Defn: **Group homomorphism**

Let $(G, *)$ and $(H, \circ)$ be two groups. A mapping $f: G \to H$ is called a group homomorphism from $(G, *)$ to $(H, \circ)$ if for any $a, b \in G$.

$$g(a * b) = g(a) \circ g(b).$$

Note: Let g be a homomorphism from $(X, *)$ to $(Y, \circ)$

(i) If $g: X \longrightarrow Y$ is $1-1$, then g is called a monomorphism

(ii) If $g: X \longrightarrow Y$ is onto, then g is called a Epimorphism

(iii) If $g: X \longrightarrow Y$ is $1-1$ and onto, then g is called isomorphism.

Example: (1)

If $*$ is a binary operation on the set R of real numbers defined by $a*b = a+b+2ab$.

(a) Find $(R, *)$ is a semigroup

(b) Find the identity element if it exists

(c) Which element has inverse and what are they?

Soln

a) (i) Let $a, b \in R \Rightarrow a*b \in R$. (Closure property)

Defined: $a*b = a+b+2ab$

$a*b \in R$, Closure Exists

(ii) Let $a, b, c \in R$ s.t $(a*b)*c = a*(b*c)$ (Associative Property)

L.H.S $(a*b)*c = (a+b+2ab)*c$

$= a+b+2ab+c+2(a+b+2ab)c$

$= a+b+2ab+c+2ac+2bc+4abc$

$= a+b+c+2ab+2ac+2bc+4abc \longrightarrow (1)$

R.H.S $a*(b*c) = a*(b+c+2bc)$

$= a+b+c+2bc+2(a)(b+c+2bc)$

$= a+b+c+2bc+2ab+2ac+4abc$

$= a+b+c+2ab+2ac+2bc+4abc \longrightarrow (2)$

L.H.S = R.H.S

i.e, $(a*b)*c = a*(b*c)$

Associative Exists

$(R, *)$ is a semigroup.

(b) To find Identity Element

Let e be an identity element

$a * e = a$

$\cancel{a} + e + 2ae = \cancel{a}$

$e + 2ae = 0$

$e(1+2a) = 0$

$e = \frac{0}{1+2a}$, if $1+2a \neq 0$

$\boxed{e = 0}$

(c) To find Inverse

Let a^{-1} be the inverse of a

$\therefore a * a^{-1} = e$

$a * a^{-1} = 0$

$a + a^{-1} + 2aa^{-1} = 0$

$a^{-1} + 2aa^{-1} = -a$

$a^{-1}(1+2a) = -a$

$a^{-1} = \frac{-a}{1+2a}$ if $1+2a \neq 0$

$2a = -1$

$a \neq -1/2$

$a^{-1} = \frac{-a}{1+2a}$, if $a \neq -1/2$

Example (2)

If $*$ is the operation defined on $S = Q \times Q$, the set of ordered pairs of rational numbers and given by $(a,b) * (x,y) = (ax, ay+b)$. Show that $(S,*)$ is a semi group. Is it commutative? Also find the identity element.

Soln

Given that $(a,b) * (x,y) = (ax, ay+b)$

(a) To prove semigroup $(S,*)$

(i) Closure property ~~a, b ∈ S~~, s.t $a*b \in S$

$\hookrightarrow (a,b) * (x,y) = (ax, ay+b)$

Closure property Exists

(ii) Associative property, $(a*b)*c = a*(b*c)$

is, $[(a,b) * (x,y)] * (c,d) = (a,b) * [(x,y) * (c,d)]$

L.H.S

$[(a,b) * (x,y)] * (c,d)$

$= (ax, ay+b) * (c,d)$

$= axc, axd + ay + b$

$= acx, adx + ay + b \longrightarrow (1)$

R.H.S $(a,b) * [(x,y) * (c,d)]$

$= (a,b) * [xc, xd+y]$

$= axc, a(xd+y) + b$

$= acx, adx + ay + b \longrightarrow (2)$

L.H.S = R.H.S

Associative law Exists

$\therefore (S,*)$ is Semi group.

(b) To find Commutative $(a*b)=(b*a)$

i, $(a,b)*(x,y) = ax, ay+b \longrightarrow (3)$

$(x,y)*(a,b) = xa, xb+y \longrightarrow (4)$

$(3) \neq (4)$

ie $(a,b)*(x,y) \neq (x,y)*(a,b)$

$\therefore$ $(S,*)$ is not commutative.

(c) To find Identity element $(a*e=a)$

Let (e_1, e_2) be the identity element of $(S,*)$, $\forall (a,b)\in S$

ie, $(a,b)*(e_1,e_2)=(a,b)$

$(ae_1, ae_2+b)=(a,b)$

$\Rightarrow ae_1=a$; $ae_2+b=b$

$e_1=a/a$; $ae_2=b-b$

$\boxed{e_1=1}$ $\quad$ $ae_2=0$

$\boxed{e_2=0}$

$\therefore$ $(1,0)$ is the identity element of $(S,*)$

Example:- (3) Show that a semi-group with more than one idempotents cannot be a group. Give an Example of a semigroup which is not a group.

Soln:

Let $(S,*)$ be semigroup

Let a, b are two idempotents

$\therefore$ $a*a=a$ & $b*b=b$.

Let us assume that $(S,*)$ is group then Each Elements has the Inverse.

$$(a*a)*a^{-1} = a*(a*a^{-1})$$

L.H.S

$(a*a)*a^{-1}$

$= a*a^{-1}$ $\{a*a=a\}$

$= e \longrightarrow (1)$

R.H.S

$a*(a*a^{-1})$

$= a*e$

$= a \longrightarrow (2)$

from (1) & (2) $\boxed{a=e}$

|||ly. We can get $\boxed{b=e}$

In a group we cannot have two identities and hence $(S,*)$ cannot be group.

This is $\Rightarrow\Leftarrow$ due to an assumption that $(S,*)$ has two idempotents.

Example: (A)

on the set Q of all rational numbers, the operation $*$ is defined by $a*b = a+b-ab$. S.T $(Q,*)$ is monoid.

Soln)

Given that $a*b = a+b-ab$

(i) Closure Property, $a,b \in Q$ s.t $a*b \in Q$

$a*b = a+b-ab$

$a*b \in Q$

Closure property Exists.

ii) Associative Property. $(a*b)*c = a*(b*c)$

L.H.S $(a*b)*c = (a+b-ab)*c$

$= a+b-ab+c-(a+b-ab)c$

$= a+b-ab+c-ac-bc+abc$

$= a+b+c-ab-ac-bc+abc \longrightarrow (1)$

R.H.S $a*(b*c) = a*(b+c-bc)$

$= a+b+c-bc-a(b+c-bc)$

$= a+b+c-bc-ab-ac+abc$

$= a+b+c-ab-ac-bc+abc \longrightarrow (2)$

L.H.S = R.H.S

Associative law Exists

iii) Identity Property, $a*e = e*a = a$

for any, $e = 0 \in a$

i, $a*e = a$

$a*0 = a+0-a(0)$

$= a-0$

$= a \longrightarrow (3)$

$e*a = a$

$0*a = 0+a-0(a)$

$= a-0$

$= a \longrightarrow (4)$

from (3) & (4) $a*e = e*a = a.$

$e = 0$ is the identity Element.

$\therefore (a,*)$ is monoid.

Chapter - 4.2

Groups & Abelian group

Theorem - ① If a and b are any two elements of a group $[G, *]$, then show that G is an abelian group if and only if $(a*b)^2 = a^2 * b^2$.

Proof:- Part-1

Given that G is an abelian group

$\Rightarrow \forall\, a, b \in G$, then $a*b = b*a \longrightarrow ①$

To Prove: $(a*b)^2 = a^2 * b^2$

$$(a*b)^2 = (a*b)(a*b)$$
$$= a*(b*a)*b \quad \text{by Eq ①}$$
$$= a*(a*b)*b$$
$$= (a*a)*(b*b)$$
$$= a^2 * b^2$$

Part-2

Given that $(a*b)^2 = a^2 * b^2$

To Prove: $a*b = b*a$

$$② \Rightarrow (a*b)^2 = a^2 * b^2$$
$$(a*b)*(a*b) = (a*a)*(b*b)$$
$$a*[b*(a*b)] = a*[a*(b*b)]$$
$$b*(a*b) = a*(b*b) \quad \text{Left Cancellation law}$$
$$(b*a)*b = (a*b)*b \quad \text{Right Cancellation law}$$
$$b*a = a*b$$

$\Rightarrow$ G is abelian group.

Theorem - (2)

If every element in a group is its own inverse, then the group must be abelian.

(OR)

For any group $(G, *)$ if $a^2 = e$ with $a \neq e$ then G is abelian.

Proof: Given that $a = a^{-1}$ ~~for~~ all $a \in G$.

Let $a, b \in G$, then $a = a^{-1}$ & $b = b^{-1}$

Now $(a*b) = (a*b)^{-1}$

ie, $a*b = b^{-1} * a^{-1}$

$= b * a$

$\therefore G$ is abelian

Theorem - (3)

The identity element of a group is unique.

Proof: Let $[G, *]$ be a group

Let e_1 and e_2 be two identity elements in G.

then $e_1 * e_2 = e_1$ $[\because e_2$ is the identity$]$

$e_1 * e_2 = e_2$ $[\because e_1$ is the identity$]$

Thus $e_1 = e_2$

Hence the identity is unique.

Theorem - 4 For any elements a in a group G, the inverse is unique.

Proof: Let 'a' be any element of a group G.

If possible, let a' & a'' be two inverses of a

then $a * a' = a' * a = e$

$a * a'' = a'' * a = e$

Now $a' = a' * e = a' * (a * a'') = (a' * a) * a'' = e * a'' = a''$

Hence, the inverse is unique.

$$(a*b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}$$
$$= a * e * a^{-1}$$
$$= a * a^{-1}$$
$$= e \longrightarrow (1)$$

& $$(b^{-1} * a^{-1}) * (a*b) = b^{-1} * (a^{-1} * a) * b$$
$$= b^{-1} * e * b$$
$$= b^{-1} * b$$
$$= e \longrightarrow (2)$$

from (1) & (2)

$$(a*b)^{-1} = b^{-1} * a^{-1}$$

Example 2 - (1) Show that $(Q^+, *)$ is an abelian group where $*$ is defined by $a*b = \frac{ab}{2}$, $\forall\, a,b \in Q^+$

Soln:

(i) Closure Property. $a,b \in Q^+$, $a*b \in Q^+$

Given that $a*b = \frac{ab}{2}$

$a*b \in Q^+$ Closure Exists

(ii) Associative Property, $(a*b)*c = a*(b*c)$

L.H.S $(a*b)*c = \frac{ab}{2}*c = \frac{\left(\frac{ab}{2}\right)c}{2}$

$= \frac{abc}{4}$

R.H.S $a*(b*c) = a*\frac{bc}{2} = \frac{(a)\left(\frac{bc}{2}\right)}{2}$

$= \frac{abc}{4}$

L.H.S = R.H.S

Associative Exists

(iii) Identity Property. $a*e = e*a = a$

Let e be an identity Element

$\therefore\; e*a = a$

$\frac{ea}{2} = a$

$e/2 = 1$

$\boxed{e=2} \in Q^+$

Identity Element Exists

(iv) Inverse Property, $a * a^{-1} = a^{-1} * a = e$

Let a^{-1} be an inverse element

$\therefore \quad a * a^{-1} = e \qquad \{ e = 2 \}$

$a * a^{-1} = 2$

$\frac{aa^{-1}}{2} = 2$

$a a^{-1} = 4$

$\boxed{a^{-1} = 4/a} \in Q^+$

Inverse element exists

(v) Commutative Property, $a, b \in Q^+$ s.t $a * b = b * a$

To Prove: $a * b = b * a$

L.H.S $a * b = \frac{ab}{2}$ & R.H.S $b * a = \frac{ba}{2} = \frac{ab}{2}$

L.H.S = R.H.S

Commutative exists.

$\therefore \ (Q^+, *)$ is Abelian group

Example :- (2)

In a group G prove that an Element $a \in G$ such that $a^2 = e$, $a \neq e$ iff $a = a^{-1}$.

Sol^n:

Let us assume that $a = a^{-1}$

then $a^2 = a * a = a * a^{-1} = e$

Conversely,

Assume that $a^2 = e$ with $a \neq e$

$$a * a = e$$

$$a^{-1}(a * a) = a^{-1} * e$$

$$(a^{-1} * a) * a = a^{-1}$$

$$e * a = a^{-1}$$

$$a = a^{-1}$$

Example-(3)

If any group $(G, *)$, show that $(a*b)^{-1} = b^{-1} * a^{-1}$

Sol^n:

Given that $(G, *)$ is a group

$\forall\, a \in G \Rightarrow a^{-1} \in G$ also $a * a^{-1} = a^{-1} * a = e$

$\forall\, b \in G \Rightarrow b^{-1} \in G$ also $b * b^{-1} = b^{-1} * b = e$

To Prove: $(a*b)^{-1} = b^{-1} * a^{-1}$

ie, To Prove: $(a*b) * (b^{-1} * a^{-1}) = (b^{-1} * a^{-1}) * (a * b) = e$.

$$(a*b) * (b^{-1} * a^{-1}) = a * (b * b^{-1}) * a^{-1}$$

$$= a * e * a^{-1}$$

$$= a * a^{-1}$$

$$= e \longrightarrow (1)$$

$$(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * b) * b$$
$$= b^{-1} * e * b$$
$$= b^{-1} * b$$
$$= e \longrightarrow (2)$$

from (1) & (2) $(a * b) * (b^{-1} * a^{-1}) = (b^{-1} * a^{-1}) * (a * b) = e$

$$\therefore (a * b)^{-1} = b^{-1} * a^{-1}$$

Example - (4)

Prove that $(Z_5, +_5)$ is abelian.

Soln:

Given that $(Z_5, +_5)$

is $Z_5 = \{0, 1, 2, 3, 4\}$

Cayley table

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) Closure Property:

from the table, closure property exists

ii) Associative Property:

clearly associative property exists

iii) Identity element:-

0 is the identity element w.r.to. $+_5$

iv) Inverse Elements:

Inverse of $0 = 0$

Inverse of $1 = 4$

Inverse of $2 = 3$

Inverse of $3 = 2$

Inverse of $4 = 1$ all the inverse elements $\in G$

v) Commutative Property:-

Clearly, addition is commutative.

Hence $(Z_5, +_5)$ is abelian group

Note: $(Z_n, +_n)$ is always abelian group

Example-(5)

Prove that $G = \{[1], [2], [3], [4]\}$ is an abelian group under multiplication modulo 5

(or)

Prove that $[Z_5^*, \times_5]$ is an abelian group

Soln: Given that $[Z_5^*, \times_5]$

is $Z_5^* = \{1, 2, 3, 4\}$

Note:- $Z_5 - \{0\} = Z_5^*$

Cayley table:-

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

i) Closure property:

from the table closure property exists

ii) Associative property:

usual multiplication is always associative

iii) Identity element:

1 is the identity element w.r.to $\times_5$

iv) Inverse elements:

Inverse of $1 = 1$

Inverse of $2 = 3$

Inverse of $3 = 2$

Inverse of $4 = 4$

all inverse elements $\in G$

v) Commutative property:

Commutative property exists under usual multiplication.

$\therefore$ G is an abelian group.

Note:

$Z_5 - (0) = Z_5^*$

$Z_5^* = \{1, 2, 3, 4\}$

$Z_3 - \{0\} = Z_3^*$

$Z_3^* = \{1, 2\}$

Example-(6)

In a group $(G, *)$, the left and right cancellation laws are true. That is $a*b = a*c \Rightarrow b = c$ and $b*a = c*a \Rightarrow b = c$.

Soln

Given that $(G, *)$ be a group

Left Cancellation law

To Prove :- $a*b = a*c \Rightarrow b = c$

Let $a*b = a*c$

Premultiply by a^{-1}

$$a^{-1}*(a*b) = a^{-1}*(a*c)$$

$$(a^{-1}*a)*b = (a^{-1}*a)*c$$

$$e*b = e*c$$

$$b = c$$

Right Cancellation law

To Prove :- $b*a = c*a \Rightarrow b = c$

Let $b*a = c*a$

Post multiply by a^{-1}

$$(b*a)*a^{-1} = (c*a)*a^{-1}$$

$$b*(a*a^{-1}) = c*(a*a^{-1})$$

$$b*e = c*e$$

$$b = c.$$

Group Homomorphism :-

Let $(G, *)$ and (H, Δ) be any two groups. A mapping $f: G \rightarrow H$ is said to be a homomorphism, if $f(a*b) = f(a) * f(b)$. for any $a, b \in G$.

Theorem-①

Homomorphism Preserves identities and inverses.

Proof:-

Let $f: (G, *) \longrightarrow (G', *)$ be a homomorphism

ie, $f(a*b) = f(a) * f(b)$.

(i) To Prove f preserves identities

ie To Prove $f(e) = e'$

Let $a \in G \Rightarrow f(a) \in G'$

$$f(a) * e' = f(a)$$
$$= f(a*e)$$
$$f(a) * e' = f(a) * f(e) \quad \{\text{left Cancellation law}\}$$
$$e' = f(e)$$

ie $f(e) = e'$

ii) To Prove f Preserves inverse

i.e. To Prove $[f(a)]^{-1} = f(a^{-1})$

We want to Prove $f(a) * f(a^{-1}) = e'$

L.H.S $= f(a) * f(a^{-1})$

$= f(a * a^{-1})$

$= f(e)$

$= e' =$ R.H.S

$f(a) * f(a^{-1}) = e'$

$\therefore [f(a)]^{-1} = f(a^{-1})$

Example. ①

Let G be a group and $a \in G$. Let $f : G \rightarrow G$ be given by $f(x) = axa^{-1}$ for all $x \in G$. Prove that f is an isomorphism of G onto G.

Soln:-

To Prove f is an isomorphism

We want to Prove:-

(i) f is one-to-one

(ii) f is onto

(iii) f is homomorphism.

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Given that $a \in G$. Let $f : G \to G$ by $f(x) = axa^{-1}$ $\forall x \in G$

(i) To find :- f is one-to-one

If $f(x) = f(y)$

then $axa^{-1} = aya^{-1}$ {by left cancellation law}

$xa^{-1} = ya^{-1}$ {by right cancellation law}

$x = y$

$\therefore$ f is one-to-one

(ii) To find :- f is onto

Let $y \in G$, then $a^{-1}ya \in G$

and $f(a^{-1}ya) = a(a^{-1}ya)a^{-1}$

$= (aa^{-1})\,y\,(aa^{-1})$

$= e \cdot (y) \cdot e$

$= y$

So, $f(x) = y$ for some $x \in G$.

f is onto

(iii) To find :- f is homomorphism

If $x, y \in G$, then

$f(x) \cdot f(y) = (axa^{-1}) \cdot (aya^{-1})$

$= ax\,(a^{-1}a)\,ya^{-1}$

$= ax\,(1)\,ya^{-1}$

$= axya^{-1}$

$= a(xy)a^{-1}$

$f(x).f(y) = f(xy)$

so f is a homorphsym.

$\therefore$ f is an isomorphism.

PERMUTATION FUNCTIONS

Defn.

A bijection from a set A to itself is Called a permutation of A.

Example :- Let $A = \{1, 2, 3\}$. Then all the permutations of A are

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

Note: $S_n = n!$

$S_2 = 2! = 2$

$S_3 = 3! = 6.$

Defn: Cyclic Permutation

Let $b_1, b_2, b_3, \ldots b_r$ be r distinct Elements of the set $A = \{a_1, a_2, a_3 \ldots a_n\}$

$P(x) = x$, if $x \in A$, $x \notin \{b_1, b_2, \ldots b_r)$ is Called a cyclic Permutation.

Defn:

Two cycles of a set A are said to be disjoint if no Element of A appears in both cycles.

Defn: Even and ODD Permutations (Transpositions)

A cycle of length 2 is called a transposition. That is, a transposition is a cycle $P = (a_i, a_j)$, where $P(a_i) = a_j$ and $P(a_j) = a_i$.

Note:

Every cycle can be written as a Product of transpositions.

Eg: (1)

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$$

cycle : $(1, 2, 4, 7)\,(3, 5, 6)$

Transposition : $(1,2)\,(1,4)\,(1,7)$

: $(3,5)\;(3,6)$

(2) Find the odd Permutation for $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 4 & 1 & 3 \end{pmatrix}$

Soln

Given that $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 4 & 1 & 3 \end{pmatrix}$

cycle: $(1, 5)\ (2, 6, 3)$

Transpositions are: $(1,5)\ (2,6)(2,3)$

The given permutation can be expressed as the product of an odd ~~permutation~~ ~~of~~ number of transpositions

$\therefore$ Hence the permutation is odd.

(3) Find the permutation is odd or Even $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

Soln

Given that $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

cycle: $(1, 6)\ (2, 3, 4, 5)$

Transpositions are: $(1,6)\ (2,3)\ (2,4)(2,5)$

No. of transpositions : 4.

Hence the permutation is Even.

(4) Find the inverse of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$

Soln

let $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$

$$P^{-1} = \begin{pmatrix} 2 & 3 & 1 & 5 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

(5) If $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$ find $AB = ?$

Soln

Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

(6) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}$ be a permutation of A. (i) Write P as a product of disjoint cycles. (ii) Compute P^{-1}, (iii) Compute P^2.

Soln

Let $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}$

(i) cycle: $(1, 2, 4)$ other elements are fixed.

(ii) To find:- P^{-1}

$$P^{-1} = \begin{pmatrix} 2 & 4 & 3 & 1 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{pmatrix}$$

To find: $P^2 = P \cdot P$

$$P \cdot P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 2 & 5 & 6 \end{pmatrix} //$$

Defn: **Isomorphism:**

Let G and G' be two groups. A map $f: G \to G'$ is said to be isomorphism if f is a bijection and f is a homomorphism.

Cayley Theorem:

statement:

Prove that Every finite of order 'n' is an isomorphic to a Permutation group of degree n.

(or)

state and Prove cayley's theorem on Permutation group.

Proof:

Let G be a group of order n.

We shall Prove this theorem by three conditions (steps)

Step-1:- First find a set G' of Permutation

Step-2:- To Prove G' is a group

Step-3:- To Prove $\phi: G \to G'$ is isomorphism

(i) Step-1 To find a Permutation set G'

Let $G' = \{ f_a / a \in G \}$

define a function $f_a : G \to G$ by $f_a(x) = ax$ where $a \in G$.

To Prove f_a is well defined.

i.e., $x = y \Rightarrow f_a(x) = f_a(y)$

let $x = y \Rightarrow ax = ay$

$f_a(x) = f_a(y)$

$\therefore$ f_a is well defined.

To Prove f_a is 1-1 (one to one)

i.e., $f_a(x) = f_a(y) \Rightarrow x = y$

let $f_a(x) = f_a(y) \Rightarrow ax = ay$ {by left cancellation}

$x = y$

$\therefore$ f_a is one-to-one

To Prove f_a is onto:

for any $y \in G$, $f_a(a^{-1}y) = a(a^{-1}y) = ey = y$

$\therefore$ f_a is onto

$\therefore$ f_a is a bijection from G to itself.

$\therefore$ f_a is a permutation.

(ii) Step-2: To Prove G' is a group

(i) Closure: let $f_a, f_b \in G'$

To Prove: $f_a \circ f_b \in G'$

Now, $f_a \circ f_b(x) = f_a[f_b(x)] = f_a(bx) = abx = f_{ab}(x)$

$\Rightarrow f_a \circ f_b(x) = f_{ab}(x)$

$\Rightarrow f_a \circ f_b = f_{ab} \longrightarrow (1)$

$\Rightarrow f_a \circ f_b \in G'$

$\therefore G'$ is Closed

(ii) Associative:

Composition is always associative

(iii) Identity:

Let e be the identity element of G.

then f_e be the identity element of G'

Identity element exists.

(iv) Inverse Element:-

Let $f_a \in G' \Rightarrow a \in G$

$\therefore a^{-1} \in G$

Now $f_a \circ f_{a^{-1}} = f_{aa^{-1}}$ [using Eqn (1)]

$= f_e$

$\therefore f_{a^{-1}}$ is the inverse of f_a

Inverse element exists

$\therefore G'$ is a group

(iii) Step-3 : Now To Prove G & G' are isomorphism

Defn: $\phi : G \to G'$ by $\phi_a = f_a$

We want to Prove :-

(i) ϕ is one-to-one

(ii) ϕ is onto

(iii) ϕ is homomorphism

(i) To Prove: ϕ is one-to-one

i.e, $\phi(a) = \phi(b) \Rightarrow a = b$

Now, $\phi(a) = \phi(b) \Longleftrightarrow f_a = f_b$

$ax = bx$ { by right ~~left~~ cancellation law }

$a = b$

$\therefore \phi$ is one-to-one

(ii) To Prove : ϕ is onto

Since, f_a is onto

$\therefore \phi_a$ is also onto

(iii) To Prove: ϕ is Homomorphism

i.e $\phi(ab) = \phi(a) \cdot \phi(b)$

Now, $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \cdot \phi(b)$

$\therefore \phi$ is a homomorphism.

Hence $\phi : G \to G'$ is an isomorphism.

Chapter - 4.3
Subgroups & Cosets

Defn **Subgroups:-**

Let $(G, *)$ be a group, then $(H, *)$ is said to be subgroup of $(G, *)$ if $H \subseteq G$ and $(H, *)$ itself is a group under the operation $*$.

(i) $e \in H$, where e is the identity in G

ii) for any $a \in H$, $a^{-1} \in H$

iii) for $a, b \in H$, $a * b^{-1} \in H$

Eg: (i) $(Q, +)$ is subgroup of $(R, +)$

(ii) $(R, +)$ is subgroup of $(C, +)$

Theorem - ①

The intersection of two subgroup of a group is also a subgroup of the group.

(OR)

Let G be a group and H_1 and H_2 are subgroups, then $H_1 \cap H_2$ is also a subgroup of G.

Proof:

Since H_1 and H_2 are subgroup of G.

$\therefore H_1 \cap H_2 \neq \phi$

let $a, b \in H_1 \cap H_2$

$a, b \in H_1$ and $a, b \in H_2$

$a * b^{-1} \in H_1$ and $a * b^{-1} \in H_2$

$a * b^{-1} \in H_1 \cap H_2$

for $a, b^{-1} \in H_1 \cap H_2$, we've $a * b^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a subgroup

Theorem (2)

The necessary and sufficient condition that a non-empty subset H of a group G to be a subgroup is $a, b \in H \Rightarrow a * b^{-1} \in H, \ \forall\, a, b^{-1} \in H$.

Proof:

Let us assume that H is a subgroup of G

Since H itself is a group

We've $a, b \in H \Rightarrow a * b \in H$

Since, $b \in H \Rightarrow b^{-1} \in H$ {H is a subgroup}

for $a, b \in H \Rightarrow a, b^{-1} \in H$

$\Rightarrow a * b^{-1} \in H$

Converse:

Let $a * b^{-1} \in H, \ \forall a, b \in H$

We want to prove H is a subgroup of G

(i) Identity: Let $a \in H \Rightarrow a^{-1} \in H$

$a * a^{-1} \in H$

$e \in H$

(ii) Inverse: Let $a, e \in H$

$a^{-1} * e \in H$

$a^{-1} \in H$

Every element 'a' of H has its inverse a^{-1} in H.

(iii) Closure: Let $b \in H \Rightarrow b^{-1} \in H$

for $a, b \in H \Rightarrow a, b^{-1} \in H$

$\Rightarrow a * (b^{-1})^{-1} \in H$

$\Rightarrow a * b \in H$

$\therefore$ H is closed.

$\therefore$ H is a subgroup of G.

Theorem - 3

The union of two subgroup of a group G is a subgroup iff one is contained in the other.

(OR)

Let H and K be two subgroup of a group G. Then $H \cup K$ is a subgroup iff either $H \subseteq K$ (or) $K \subseteq H$.

Proof

Let us assume H and K be are two subgroup of G.

and $H \subseteq K$ (or) $K \subseteq H$

$\therefore H \cup K = K$ (or) $H \cup K = H$

Hence $H \cup K$ is a subgroup.

Converse:

Suppose $H \cup K$ is a subgroup of G.

we claim that $H \subseteq K$ (or) $K \subseteq H$

Suppose that, $H \not\subseteq K$ or) $K \not\subseteq H$

$\exists$ element a, b such that

$a \in H$ and $a \notin K \longrightarrow (1)$

$b \in K$ and $b \notin H \longrightarrow (2)$

Clearly, $a, b \in H \cup K$. Since $H \cup K$ is subgroup

$ab \in H \cup K$.

Hence $ab \in H$ (or) $ab \in K$

<u>Case-(i)</u> Let $ab \in H$, since $a \in H$, $a^{-1} \in H$

$a^{-1}(ab) \in H$

$(a^{-1}a)b \in H$

$b \in H$

Which is $\Rightarrow\Leftarrow$ to (2)

<u>Case-(ii)</u> Let $ab \in K$, Since $b \in K$, $b^{-1} \in K$.

$b^{-1}(ab) \in K$

$b^{-1}(ba) \in K$

$(b^{-1}b)a \in K$

$a \in K$

Which is $\Rightarrow\Leftarrow$, to (1)

$\therefore$ our assumption is wrong.

$\therefore$ $H \subseteq K$ (or) $K \subseteq K$

<u>Example-(1)</u> The identity element of a subgroup is same as that of the group.

<u>Soln:</u>

Let H be the subgroup of the group G.

then e and e' be the identity elements of G & H

Now, if $a \in H$, then $a \in G$ ~~& ae~~

and $a * e = a$ because e is the identity Element of G

Again, $a \in H$, then $a * e' = a$, since e' is the identity Element of H.

Thus $a * e = a * e'$ {by left Cancelling law}

$$e = e'$$

$\therefore$ The ~~Elements~~ identity Element of a subgroup is same as the group.

<u>Example-(2)</u> Is the union of two subgroup of a group, a subgroup of G? Justify your answer.

<u>Soln:</u> The union of two subgroup of a group need not be a subgroup of G.

Let the group $(\mathbb{Z}, +)$

Let $H = 3\mathbb{Z} = \{0, \pm 3, \pm 6 - --\}$ & let $K = 2\mathbb{Z} = \{0, \pm 2, \pm 4, --\}$

$\Rightarrow$ H and K are subgroup of $\{\mathbb{Z}, +\}$

$\Rightarrow$ $3 \in 3\mathbb{Z} \in 3\mathbb{Z} \cup 2\mathbb{Z} = H \cup K$.

$\Rightarrow$ $2 \in 2\mathbb{Z} \in 3\mathbb{Z} \cup 2\mathbb{Z} = H \cup K$.

But $3 + 2 = 5 \notin 3\mathbb{Z} \cup 2\mathbb{Z}$

$\therefore$ $H \cup K$ is not a subgroup of $(\mathbb{Z}, +)$

Example - (3)

Show that the kernel of a homomorphism of a group $[G, *]$ into another group $[H, \circ]$ is a subgroup of G.

Soln:- Let K be the kernel of the homomorphism g

That is $K = \{x \in G \,/\, g(x) = e'\}$

where e' is the identity element of H.

Let $x, y \in K$.

Now $g(x * y^{-1}) = g(x) \circ g(y^{-1})$

$$= g(x) \circ [g(y)]^{-1}$$

$$= e' \circ (e')^{-1}$$

$$= e' \circ e'$$

$$= e'$$

$\therefore x * y^{-1} \in K$.

Therefore K is a subgroup of G.

Example - (4)

Find all the subgroup of $(Z_9, +_9)$

Soln: Let $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

then subgroup generator by

$\langle 0 \rangle = \{0\}$

$\langle 1 \rangle = \{1, 2, 3, 4, 5, 6, 7, 8, 0\}$

$\langle 2 \rangle = \{2, 4, 6, 8, 1, 3, 5, 7, 0\}$

$\langle 3 \rangle = \{3, 6, 0\}$

$\langle 4 \rangle = \{4, 8, 3, 7, 2, 6, 1, 5, 0\}$

$\langle 5 \rangle = \{5, 1, 6, 2, 7, 3, 8, 4, 0\}$

$\langle 6 \rangle = \{6, 3, 0\}$

$\langle 7 \rangle = \{7, 5, 3, 1, 8, 6, 4, 2, 0\}$

$\langle 8 \rangle = \{8, 7, 6, 5, 4, 3, 2, 1, 0\}$

$\therefore$ subgroup of Z_9 are $\{0\}$, $\{0, 3, 6\}$, Z_9

Improper subgroup $= \{0\}, Z_9$

Proper subgroup $= \{0, 3, 6\}$.

Example-5 Show that the set of all elements a of a group $(G, *)$ such that $a * x = x * a$ for every $x \in G$ is a subgroup.

Soln

let $H = \{a \in G \,/\, a * x = x * a, \; \forall x \in G\}$

(i) here, $e * x = x * e = x$

$\therefore e \in H$

H is non empty.

ii) let $a, b \in H \Rightarrow a * x = x * a$ & $b * x = x * b$
$\hookrightarrow (1)$ $\hookrightarrow (2)$

To Prove: $(a * b^{-1}) \in H$

We want Prove: $(a * b^{-1}) * x = x * (a * b^{-1})$

L.H.S $(a * b^{-1}) * x$

$= a * (b^{-1} * x)$

$= a * (x * b^{-1})$

$= (a * x) * b^{-1}$

$= (x * a) * b^{-1}$

$= x * (a * b^{-1})$

$=$ R.H.S

$\therefore (a * b^{-1}) \in H$.

$\therefore$ H is a subgroup of G.

Defn: Cosets

(i) Left Coset of H in G

Let $(H, *)$ be a subgroup of $(G, *)$ for any $a \in G$, the left coset of H, denoted by $a * H$ is the set $a * H = \{a * h / h \in H\}$ $\forall a \in G$.

ii) Right Coset of H in G.

The right coset of H, denoted by $H * a$ is the set $H * a = \{h * a / h \in H\}$, $\forall a \in G$.

Eg: ① Let $G = \{1, a, a^2, a^3\}$ $(a^4 = 1)$ be a group and $H = \{1, a^2\}$ is a subgroup of G under multiplication. Find all the coset of H.

Soln:

The right Cosets of H in G.

Given that $G = \{1, a, a^2, a^3\}$ & $H = \{1, a^2\}$

$H * 1 = \{1, a^2\} * 1 = \{1, a^2\} = H$

$H * a = \{1, a^2\} * a = \{a, a^3\} = \{1, a^2\} a = H * a$

$H * a^2 = \{1, a^2\} * a^2 = \{a^2, a^4\} = \{a^2, 1\} = \{1, a^2\} = H$

$H * a^3 = \{1, a^2\} * a^3 = \{a^3, a^5\} = \{a^3, a\} = \{a^2, 1\} a = H * a$

Subgroups:

$\therefore \quad H*1 = H = H*a^2 = \{1, a^2\}$

$H*a = H*a^3 = \{a, a^3\}$ are two distinct right cosets of H in G.

Illy, we can find the left coset of H in G.

(2) Consider the group $Z_4 = \{[0], [1], [2], [3]\}$, of integers modulo 4. Let $H = \{[0], [2]\}$ be a subgroup of Z_4 under $+_4$ (addition mod 4).

Soln:

Given that $Z_4 = \{[0], [1], [2], [3]\}$ & $H = \{[0], [2]\}$

The left cosets of H are,

~~[0] + [H] =~~

$[0] + H = [0] + \{[0], [2]\} = \{[0], [2]\} = H$

$[1] + H = [1] + \{[0], [2]\} = \{[1], [3]\} = [1] + H$

$[2] + H = [2] + \{[0], [2]\} = \{[2], [4]\} = \{[2], [0]\} = H$

$[3] + H = [3] + \{[0], [2]\} = \{[3], [5]\} = \{[3], [1]\} = [1] + H$

$\therefore \quad [0] + H = [2] + H = H$

$[1] + H = [3] + H = [1] + H$ are two distinct left cosets of H in Z_4.

Theorem: Any two right or left cosets of H in G are Either disjoint or identical.

Proof: Let $H*a$ and $H*b$ be the two right cosets of a subgroup H of G.

Let $a, b \in G$.

We've to Prove that Either

$(H*a) \cap (H*b) = \phi$ (or) $H*a = H*b$

Suppose, $(H*a) \cap (H*b) \neq \phi$, then there exists an element $x \in (H*a) \cap (H*b)$

$\Rightarrow x \in H*a$ & $x \in H*b$

Now $x \in H*a$ and $x \in H*b$

$\Rightarrow H*x = H*a \rightarrow (1)$ $\quad \Rightarrow H*x = H*b \rightarrow (2)$

from (1) & (2), we've

$H*x = H*a = H*b$

$\therefore H*a = H*b$

Either, $H*a \cap H*b = \phi$ (or) $H*a = H*b$.

Lagrange's Theorem:-

Let G be a finite group of order 'n' and H be any subgroup of G. Then the order of H divides the order of G. ie, $o(H) / o(G)$.

(or)

The order of each subgroup of a finite group is a divisor of the order of the group.

Proof:- Let $(G, *)$ be a group, whose order is 'n'

$$o(G) = n$$

Let $(H, *)$ be a subgroup of G, where order is 'm'

$$o(H) = m.$$

Let $h_1, h_2, h_3, \ldots \ldots h_m$ be the 'm' different elements of H

The right coset $H*a$ of H in G is defined by

$$H*a = \{h_1*a, h_2*a, \ldots \ldots h_m*a\}$$

Since, there is a 1-1 correspondence between the elements of H and $H*a$

the elements of $H*a$ are distinct

Hence each right coset of H in G has m distinct elements

The no. of distinct right cosets of H in G is finite

The union of these k distinct cosets of H in G is equal to G.

Let these k distinct right cosets be

$$H * a_1, H * a_2, \ldots H * a_k$$

then $G = (H * a_1) \cup (H * a_2) \cup \ldots \cup (H * a_k)$

$$o(G) = o(H * a_1) + o(H * a_2) + \ldots + o(H * a_k)$$

$$n = m + m + m + \ldots + m \,(k \text{ times})$$

$$n = km$$

$$\frac{n}{m} = k.$$

ie $\dfrac{o(G)}{o(H)} = k.$

$\Rightarrow o(H)$ divides $o(G)$.

Chapter - 4.4

Normal subgroup & Cyclic group

Defn: Normal subgroup

If H is said to be a normal subgroup of G. for every $x \in G$ and for $h \in H$, if $x * h * x^{-1} \in H$, $x * H * x^{-1} \subseteq H$.

(OR)

A subgroup H of G is called a normal subgroup of G if $x * h = h * x$, $\forall\ x \in G$.

Defn: Cyclic group

A group $(G, *)$ is said to be cyclic if there exists an element $a \in G$ such that every element of G can be written as some power of 'a'.

Theorem - ①

The intersection of any two normal subgroups of a group is a normal subgroup.

(or)

If H and K are normal subgroup of a group G, then $H \cap K$ is also a normal subgroup.

Proof:

Given that H and K are normal subgroups

$\Rightarrow$ H and K are subgroup of G

$\Rightarrow$ $H \cap K$ is a subgroup of G

Now, we've to prove that $H \cap K$ is normal

let $x \in G$ and $h \in H \cap K$

$x \in G$ and $h \in H$ & $h \in K$

$x \in G$, $h \in H$ and $x \in G$, $h \in K$

$x * h * x^{-1} \in H$, $x * h * x^{-1} \in K$.
$\hookrightarrow ①$ $\hookrightarrow ②$

{∵ H and K are normal subgroup}

from ① & ②, we get

$x * h * x^{-1} \in H \cap K$

$\Rightarrow$ $H \cap K$ is a normal subgroup of G.

Theorem:- ②

Every subgroup of an abelian group is a normal subgroup.

Proof:-

Let $(G, *)$ be an abelian group and $(N, *)$ be an subgroup of G.

Let g be any element in G and let $n \in N$

Now,
$$\begin{aligned} g * n * g^{-1} &= n * g * g^{-1} \\ &= n * (g * g^{-1}) \quad \{G \text{ is abelian}\} \\ &= n * e \\ &= n \in N. \end{aligned}$$

$\therefore \forall g \in G$ and $n \in N$, $g * n * g^{-1} \in N$

$\therefore (N, *)$ is a normal subgroup

Theorem:- (3)

Prove that Every cyclic group is an abelian group.

Proof:

Let $(G, *)$ be the cyclic group generated by an Element $a \in G$.

then for any two element $x, y \in G$.

we've $x = a^n$, $y \in a^m$, where m, n are integer.

Now $x * y = a^n * a^m$

$$= a^{n+m}$$

$$= a^{m+n}$$

$$= a^m * a^n$$

$$x * y = y * x$$

Hence $(G, *)$ is abelian.

Theorem:- (4)

Prove that Every subgroup of a cyclic group is normal.

Soln

We know that Every cyclic group is abelian

ie, $x * y = y * x$

Let G be the cyclic group and let H be a subgroup of G

Let $x \in G$ and $h \in H$, then

$$x * h * x^{-1} = x * (h * x^{-1})$$

$$= x * (x^{-1} * h)$$

$$= (x * x^{-1}) * h$$

$$= e * h$$

$$= h \in H$$

Thus for $x \in G$ and $h \in H$, $x * h * x^{-1} \in H$

ie, H is a normal subgroup of G.

$\therefore$ Every subgroup of a cyclic group is normal.

Theorem:- (5)

Let $f: G \rightarrow G'$ be a homomorphism, then the $\ker(f)$ is normal subgroup of G.

(or)

Let $(G, *)$ and (H, Δ) be groups and $f: G \rightarrow H$ be a homomorphism, then the kernal of f is a normal subgroup.

Proof:-

Let K be the kernel of the homomorphism f.

ie, $K = \{x \in G \,/\, f(x) = e'\}$.

where e' the identity element of H

Let $x, y \in K$, Now

$$
\begin{aligned}
f(x * y^{-1}) &= f(x) \Delta f(y^{-1}) \\
&= f(x) \Delta [f(y)]^{-1} \\
&= e' \Delta (e')^{-1} \\
&= e' \Delta e' \\
&= e'
\end{aligned}
$$

ie, $x * y^{-1} \in K$.

$\therefore$ K is a subgroup of G.

Let $x \in K$, $h \in G$

$$f(h * x * h^{-1}) = f(h) * f(x) * f(h^{-1})$$
$$= f(h) * e * [f(h)]^{-1}$$
$$= f(h) * [f(h)]^{-1}$$
$$= e'$$

i.e., $h * x * h^{-1} \in K$.

$\therefore$ K is a normal subgroup of G.

Theorem: (6)

Let $(H, *)$ be a subgroup of $(G, *)$. Then show that $(H, *)$ is a normal subgroup if and only if $x * H * x^{-1} = H$, $\forall x \in G$.

(or)

Prove that a subgroup H of a group is normal iff $x * H * x^{-1} = H$, $\forall x \in G$.

Proof:

Part-1

Let H be a normal in G

then by definition $x * H = H * x$, $\forall x \in G$

i.e., $x * H * x^{-1} = x * (x^{-1} * H)$

$$= (x * x^{-1}) * H$$
$$= e * H$$

$$x * H * x^{-1} = H$$

Part-2

Let $x^{-1} * H * x = H, \ \forall x \in G$

Permultiply by 'x'

$$x*(x^{-1} * H * x) = x * H$$

$$(x * x^{-1}) * (H * x) = x * H$$

$$e * (H * x) = x * H$$

$$H * x = x * H$$

$\therefore$ H is a normal subgroup

Theorem:- (4)

Show that every subgroup of a cyclic group is cyclic

Soln.

Let $(G, *)$ be a cyclic group generated by an element 'a'

$\Rightarrow$ Every elements in G of the form a^n, n is integer

Let H be a subgroup of G.

To prove H is cyclic

Case-(i) If H is an improper subgroup

ie, $H = \{e\}$, $H = G$

then obviously H is cyclic subgroup

Case-(ii) If H is proper subgroup ie, $(H \subset G)$

Let $H = \{a^m, a^s, a^t, \ldots\}$

To prove H is cyclic

Let m be the smallest positive integer,

such that $a^m \in H$

Now to prove H is cyclic group generated by a^m

ie, To prove $H = \langle a^m \rangle$

Let $x \in H \Rightarrow x = a^k$, k is an integer

by division algorithm

$$k = mq + r, \quad 0 \leq r < m$$

$$a^k = a^{mq+r}$$

$$a^k = a^{mq} \cdot a^r$$

$$\frac{a^k}{a^{mq}} = a^r$$

$$(a^k)(a^{mq})^{-1} = a^r$$

ie, $a^r = (a^k) \cdot (a^{mq})^{-1}$

$$a^r \in H$$

$\Rightarrow \boxed{r = 0}$, $\therefore k = mq$

$$x = a^k = a^{mq} = (a^m)^q$$

$\therefore$ H is cyclic subgroup

Example:
show that the group $G = \{1, -1, i, -i\}$ is cyclic and find its generators.

Soln: Given that $G = \{1, -1, i, -i\}$

To Prove G is cyclic:-

we want to find the generator

(i) Let $a = i$, then

$$\langle a \rangle = \langle i \rangle = \{(i)^1, (i)^2, (i)^3, (i)^4\}$$
$$= \{i, -1, -i, 1\}$$
$$= \{1, -1, i, -i\} = G$$

$$\left\{ \begin{array}{l} i^2 = -1 \\ i^3 = (i)(i)^2 = -i \\ i^4 = (i^2)(i^2) = 1 \end{array} \right.$$

(ii) Let $a = -i$, then

$$\langle a \rangle = \langle -i \rangle = \{(-i)^1, (-i)^2, (-i)^3, (-i)^4\}$$
$$= \{-i, 1, i, -1\}$$
$$= \{1, -1, i, -i\} = G$$

Here, i and $-i$ are generator of G.

$\therefore$ G is a cyclic group.

Defn: Ring

An non-Empty set $(R, +, \cdot)$ is called a ring if the binary operations $+$ and $\cdot$ satisfies the following conditions.

(i) $(a+b)+c = a+(b+c)$, $\forall a,b,c \in R$

(ii) $a+0 = 0+a = a$, $\forall a \in R$

(iii) $a+(-a) = (-a)+a = 0$, $\forall a, a^{-1} \in R$

(iv) $a+b = b+a$, $\forall a,b \in R$

(v) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall a,b,c \in R$

(vi) The operation $\cdot$ is distributive over $+$

ie, $a \cdot (b+c) = a \cdot b + a \cdot c$

$(b+c) \cdot a = b \cdot a + c \cdot a$

Eg: $(Z, +, \cdot)$, $(Q, +, \cdot)$, $(R, +, \cdot)$, $(C, +, \cdot)$

Defn: Commutative Ring

The Ring $(R, +, \cdot)$ is called Commutative Ring, if $ab = ba$ for $a, b \in R$.

Defn: If $(R, \cdot)$ is monoid, then the ring $(R, +, \cdot)$ is called a ring with identity (or) unity.

Defn <u>Zero divisors</u>

If $a \neq 0$, $b \neq 0$, elements of a ring R, such that $a \cdot b = 0$, then a & b are called zero divisors.

Defn. <u>Integral domain</u>

A commutative ring $(R, +, \cdot)$ with identity and without zero divisors is called an integral domain.

Eg: $(Z, +, \cdot)$ is an integral domain

Defn <u>Field</u>

A commutative ring with identity $(R, +, \cdot)$ is called a field, if every non-zero elements has a multiplicative inverse.

Thus $(R, +, \cdot)$ is a field.

i) $(R, +)$ is abelian group

ii) $\{R - \{0\}, \cdot\}$ is abelian group

Eg: $(R, +, \cdot)$ is a field.

$(Q, +, \cdot)$ is a field.

Example:

Prove that the set $Z_4 = \{[0], [1], [2], [3]\}$ is a Commutative ring with respect to the binary operation addition modulo and multiplication modulo $+_4$, $\times_4$.

Soln:

Given that $Z_4 = \{[0], [1], [2], [3]\}$

Cayley table for additive modulo 4. (i.e $+_4$)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Cayley table for multiplicative modulo 4 (i.e $\times_4$)

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(i) All the Entries in both the table belong to Z_4

$\therefore$ Z_4 closed under the operation $+_4$ and $\times_4$

ii) Also, for any $a, b, c \in Z_4$, we have

$$a + (b + c) = (a + b) + c$$

and $$a \times (b \times c) = (a \times b) \times c$$

Since, $0+(1+2) = (0+1)+2$

$0+3 = 1+2$

$3 = 3$

also $1\times(2\times3) = (1\times2)\times3$

$1\times6 = 2\times3$

$6 = 6$

Thus, the operations $+_4$ and $\times_4$ are associative in Z_4.

(iii) 0 is the additive identity of Z_4 and

1 is the multiplicative identity of Z_4.

(iv) Additive inverse of 0, 1, 2, 3 are 0, 3, 2, 1 respectively

multiplicative inverse of 1, 2, 3 are 1, 2, 3 respectively

(v) In both the tables

Entries in the first row = Entries in the first column

Entries in the second row = Entries in the second column

Entries in the third row = Entries in the third column

Entries in the fourth row = Entries in the fourth column

$\therefore$ The operations $+_4$ and $\times_5$ are commutative in Z_4.

(vi) If $a, b, c \in Z_4$ then

$a\times(b+c) = (a\times b)+(a\times c)$

and $(a+b)\times c = (a\times c)+(b\times c)$

Thus, the operation $\times_4$ is distributed over $+_4$ in Z_4

Hence, $(Z_4, +_4, \times_4)$ is a commutative ring with unity.

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2024–2025

MA3354-
DISCRETE MATHEMATICS

UNIT-5

LATTICES AND BOOLEAN ALGEBRA

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EFFECTIVE DATE:06.10.2017

UNIT-5

LATTICES AND BOOLEAN ALGEBRA

Partial ordering – Posets – Lattices as posets – Properties of lattices - Lattices as algebraic systems – Sub lattices – Direct product and homomorphism – Some special lattices – Boolean algebra – Sub Boolean Algebra – Boolean Homomorphism

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UNIT-5

LATTICES AND BOOLEAN ALGEBRA

Chapter - 5.1 [Lattices]

Defn: Relation:

A relation R on a set x is a well defined rule, if x and y are related then we write xRy if not $x \not R y$.

Defn: Partial order Relation

A relation R on a set A is said to be partial order relation, if

(i) R is reflexive $xRx, \forall x \in A$

(ii) R is antisymmetric if xRy & $yRx \Rightarrow x = y, \forall x, y \in A$

(iii) R is transitive if $xRy, yRz \Rightarrow xRz, \forall x, y, z \in A$

Defn: Partially order set (or) Poset

A set P together with a partial ordering R is called a Partially order set (or) a Poset.

~~Eg:~~ usually partial order set is denoted by '$\leq$' ie $(P, \leq)$

Eg: $Z_+ = \{1, 2, 3, 4, \ldots \infty\}$ & $R \rightarrow /$ ÷

(i) Reflexive :- clearly $x/x, \forall x \in Z^+$

(ii) Antisymmetric :- if x/y & $y/x \Rightarrow x = y$

(iii) Transitive : if x/y & $y/z \Rightarrow x/z$

$\therefore (Z_+, /)$ is Poset.

Defn: Totally ordered set (or) chain

Let $(X, \leq)$ be a Poset. We say that $(X, \leq)$ is a chain if every two elements of X are comparable

i.e., $\forall\, a, b \in X$, either $a \leq b$ (or) $b \leq a$.

Eg: $(\mathbb{Z}, \leq)$ is chain

$2 \leq 3$ (or) $3 \leq 2$.

Defn: Upper bound and Lower bound

Let $(P, \leq)$ be a Poset and A be a subset of P.

* An element $a \in P$ is said to be upper bound for A if $a \geq x, \forall\, x \in A$
* An element $b \in P$ is said to be Lower bound for A if $b \leq x, \forall\, x \in A$.

Defn: Least upper bound (LUB)

Let $(P, \leq)$ be a Poset and $A \subseteq P$ an element $a \in P$ is said to be LUB of A if

(i) $a \geq x, \forall\, x \in A$

(ii) if c is any other upper bound of A then $a \leq c$.

Defn: Greatest Lower bound (GLB)

Let $(P, \leq)$ be a Poset and $A \subseteq P$ we say that $a \in P$ is said to be GLB of A if

(i) $a \leq x, \forall x \in A$

(ii) if $b \in P$ is Lower bound of A then $b \leq a$.

Defn: Hasse Diagram

Hasse diagram of a finite Partially ordered Set S is the directed graph whose vertices are the elements of S and there is a directed Edge from a to b whenever $a \leq b$ in S.

Ex: In the Poset $(Z^+, /)$ are the integers 3 and 9 Comparable? Are 5 and 7 Comparable?

Soln:

Let $(Z^+, /)$ is Poset.

Since $3/9$, the integers 3 and 9 are Comparable.

for 5, 7 neither $5/7$ nor $7/5$

$\therefore$ 5 and 7 are not Comparable

Defn: G.L.B and L.U.B

Soln:

$$GLB\{a, b\} = a * b$$

$$LUB\{a, b\} = a \oplus b$$

Problem-①

If (A, R) is a Partially ordered set then show that the set (A, R^{-1}) is also a partially ordered set where $R^{-1} = \{(b,a) / (a,b) \in R\}$.

Proof:- To Prove:- (A, R^{-1}) is a Poset.

(i) Reflexive:- Let $a \in A$

Given that (A, R) is a poset

$\Rightarrow R$ is Reflexive $(a,a) \in R$

$\Rightarrow (a,a) \in R^{-1}$

$\therefore (A, R^{-1})$ is Reflexive.

(ii) Antisymmetric:-

If $(a,b) \in R^{-1}$ & $(b,a) \in R^{-1}$

then to Prove $a=b$

$\Rightarrow (b,a) \in R$ & $(a,b) \in R$

Since R is Antisymmetric then $a=b$

(iii) Transitive:-

Let $a,b,c \in A$ such that $(a,b) \in R^{-1}$ & $(b,c) \in R^{-1}$

then To Prove $(a,c) \in R^{-1}$

$\Rightarrow (b,a) \in R$ & $(c,b) \in R$

$c R b R a$

Since, $R \rightarrow$ antisymmetric $(c,a) \in R \Rightarrow (a,c) \in R^{-1}$

Defn: **Hasse Diagram**

Pictorial representation of a Poset is called Hasse diagram.

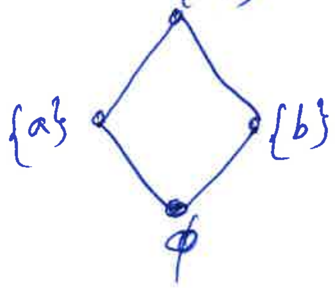
Eg: ① Draw Hasse diagram for $[P(A), \subseteq]$, $A = \{a, b\}$

Soln:

Let $A = \{a, b\}$

Hasse diagram for $[P(A), \subseteq]$

$P(A) = [\{a\}, \{b\}, \{a, b\}, \phi]$



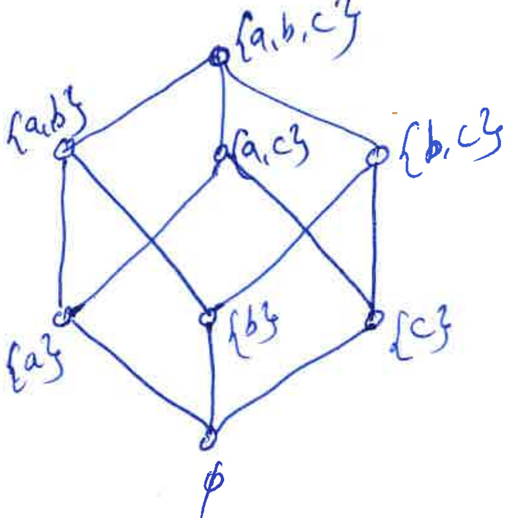
Eg - ② Draw Hasse diagram for $[P(A), \subseteq]$, $A = \{a, b, c\}$

Soln:

Let $A = \{a, b, c\}$

Hasse diagram for $[P(A), \subseteq]$

$P(A) = [\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \phi]$



Example - ①

Consider $X = \{1,2,3,4,6,12\}$, $R = \{(a,b) / a|b\}$

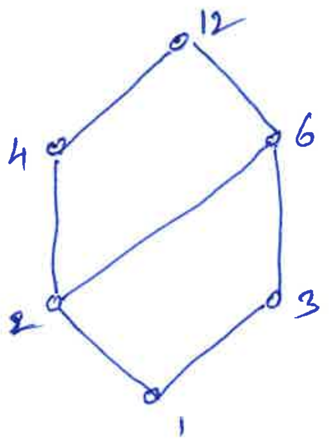
Find LUB and GLB for the Poset (X, R)

Soln

Let $X = \{1,2,3,4,6,12\}$

then the Relation $R = \left\{ \begin{array}{l} (1,2),(1,3),(1,4),(1,6),(1,12),(2,4) \\ (2,6),(2,12),(3,6),(3,12),(4,12) \end{array} \right\}$

The Hasse diagram for (X, R) is



UPPER BOUND	LOWER BOUND
1. $UB\{1,3\} = \{3,6,12\}$	1. $LB\{1,3\} = 1$
$LUB\{1,3\} = 3$	$GLB\{1,3\} = 1$
2. $UB\{1,2\} = \{2,4,6,12\}$	2. $LB\{1,2\} = 1$
$LUB\{1,2\} = 2$	$GLB\{1,2\} = 1$
3. $UB\{1,2,3\} = \{6,12\}$	3. $LB\{1,2,3\} = 1$
$LUB\{1,2,3\} = 6$	$GLB\{1,2,3\} = 1$
4. $UB\{2,3\} = \{6,12\}$	4. $LB\{2,3\} = 1$
$LUB\{2,3\} = 6$	$GLB\{2,3\} = 1$
5. $UB\{2,3,6\} = \{6,12\}$	5. $LB\{2,3,6\} = 1$
$LUB\{2,3,6\} = 6$	$GLB\{2,3,6\} = 1$
6. $UB\{4,6\} = \{12\}$	6. $LB\{4,6\} = \{1,2\}$
	$GLB\{4,6\} = 2.$

Example - (2)

Let $D_{30} = \{1, 2, 3, 5, 6, 15, 30\}$ and let the relation R be divisor on D_{30}. Find

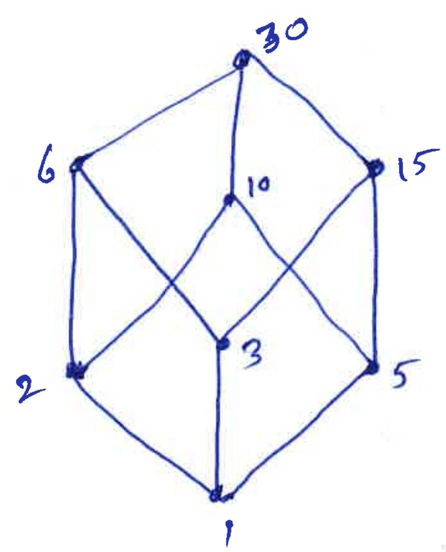
i) Draw the Hasse diagram
ii) All the lower bounds of 10 and 15
iii) the GLB of 10 and 15
iv) All the upper bounds of 10 and 15
v) the LUB of 10 and 15

Soln:

Let $D_{30} = \{1, 2, 3, 5, 6, 15, 30\}$

then the Relation $= \{(1,2), (1,3), (1,5), (2,10), (2,6), (3,6), (3,15), (5,10), (5,15), (6,30), (10,30), (15,30)\}$

(i) The Hasse diagram for (D_{30}, R) is



ii) The lower bounds of 10 & 15

$LB\{10\} = \{10, \underline{5}, 2, \underline{1}\}$

$LB\{15\} = \{15, \underline{5}, 3, \underline{1}\}$

$LB\{10, 15\} = \{1, 5\}$

iii) The GLB of 10 & 15

$$GLB\{10, 15\} = \{5\}$$

iv) The upper bounds of 10 & 15

$$UB\{10\} = \{10, \underline{30}\}$$

$$UB\{15\} = \{15, \underline{30}\}$$

$$UB\{10, 15\} = \{30\}$$

v) The LUB of 10 & 15

$$LUB\{10, 15\} = \{30\}$$

<u>Example-⑤</u>

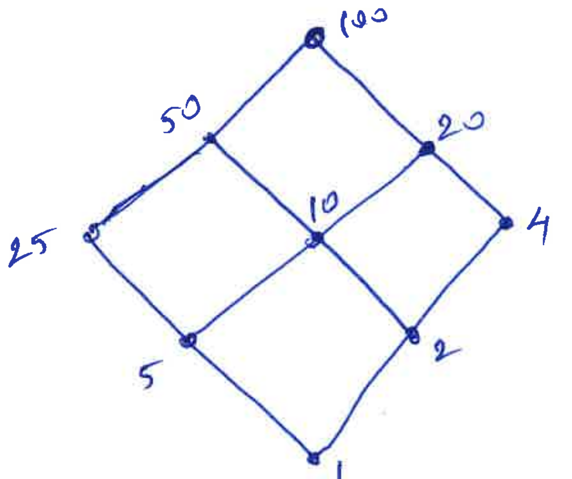
Let $D_{100} = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ be the divisions of 100. Draw the Hasse diagram of $(D_{100}, /)$ where / is the relation "division". Find the

(i) $GLB\{10, 20\}$, (ii) $LUB\{10, 20\}$, (iii) $GLB\{5, 10, 20, 25\}$

iv) $LUB\{5, 10, 20, 25\}$.

<u>Soln:</u> Let $D_{100} = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$

The Hasse diagram of $(D_{100}, /)$



i) To find $GLB\{10, 20\}$

$LB\{10\} = \{10, 5, 2, 1\}$

$LB\{20\} = \{20, 10, 5, 4, 2, 1\}$

$LB\{10, 20\} = \{10, 5, 2, 1\}$

$GLB\{10, 20\} = \{10\}$

ii) To find $LUB\{10, 20\}$

$UB\{10\} = \{10, 20, 50, 100\}$

$UB\{20\} = \{20, 100\}$

$UB\{10, 20\} = \{20, 100\}$

$LUB\{10, 20\} = \{20\}$

iii) To find $GLB\{5, 10, 20, 25\}$

$LB\{5\} = \{5, 1\}$

$LB\{10\} = \{10, 5, 2, 1\}$

$LB\{20\} = \{20, 10, 5, 4, 2, 1\}$

$LB\{25\} = \{25, 5, 1\}$

$LB\{5, 10, 20, 25\} = \{5, 1\}$

$GLB\{5, 10, 20, 25\} = \{5\}$

iv) To find $LUB\{5, 10, 20, 25\}$

$UB\{5\} = \{5, 10, 20, 25, 50, 100\}$

$UB\{10\} = \{10, 20, 50, 100\}$

$UB\{20\} = \{20, 100\}$

$UB\{25\} = \{25, 50, 100\}$

$UB\{5, 10, 20, 25\} = \{100\}$

$LUB\{5, 10, 20, 25\} = \{100\}$

Example-④ Consider the set $D_{50} = \{1, 2, 5, 10, 25, 50\}$ and the relation (/ divisor) be a partial order relation on D_{50}

(i) Draw the Hasse diagram of D_{50}

ii) Determine all the upper bound of 5 & 10

iii) Determine the LUB of $\{5, 10\}$

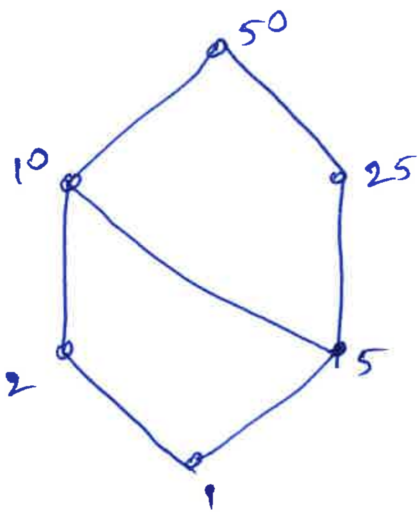
iv) Determine all the lower bound of 5 & 10

v) Determine the GLB of $\{5, 10\}$

Soln

Let $D_{50} = \{1, 2, 5, 10, 25, 50\}$

(i) The Hasse diagram for D_{50}



ii) The upper bound of 5 & 10

$UB\{5\} = \{5, \underline{10}, 25, \underline{50}\}$

$UB\{10\} = \{\underline{10}, \underline{50}\}$

$UB\{5, 10\} = \{10, 50\}$

iii) The LUB of $\{5, 10\}$

$LUB\{5, 10\} = \{10\}$

iv) The Lower bound of 5 & 10

$LB\{5\} = \{\underline{5}, \underline{1}\}$

$LB\{10\} = \{10, \underline{5}, 2, \underline{1}\}$

$LB\{5, 10\} = \{5, 1\}$

v) The GLB of $\{5, 10\}$

$GLB\{5, 10\} = \{5\}$

<u>Example-5</u>

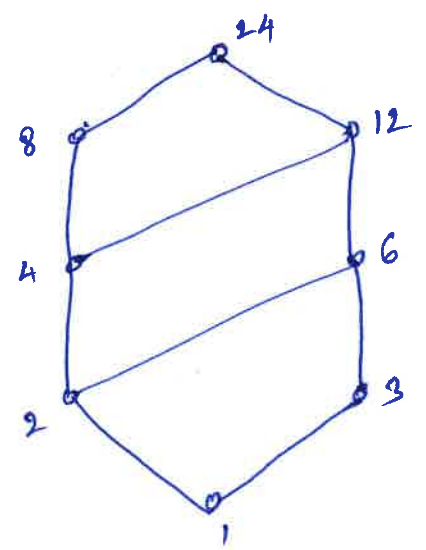
Draw the Hasse diagram for

(i) $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

ii) $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

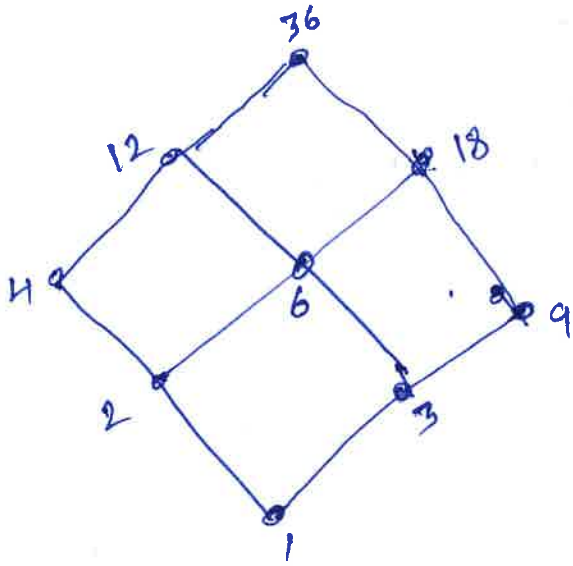
<u>Soln:</u>

i) $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



$(D_{24}, /)$

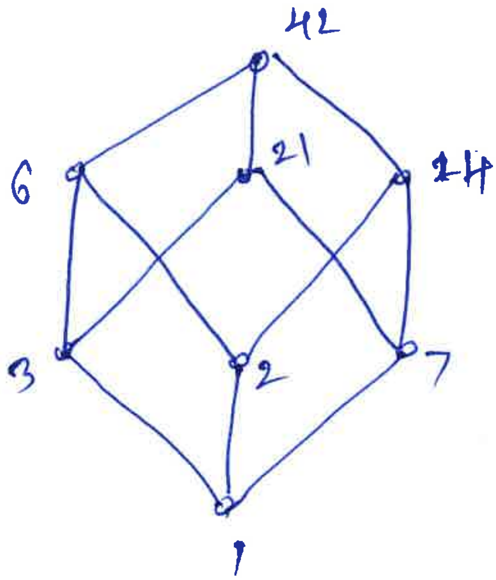
ii) $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



Example-6 Draw a Hasse diagram for $D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$.

Soln let $D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$

The Hasse diagram of $(D_{42}, /)$



Defn: Lattice

A Lattice is a Poset $(L, \leq)$ in which for Every Pair of Elements $a, b \in L$, both GLB and LUB Exists.

Note:

$$GLB\{a,b\} = a * b \text{ (or) } a \wedge b \text{ (or) } a \cdot b$$

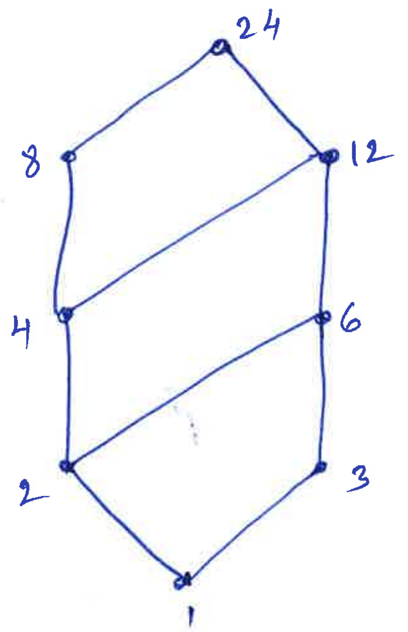
$$* \text{ (or) } \wedge \rightarrow \text{meet}$$

$$LUB\{a,b\} = a \oplus b \text{ (or) } a \vee b \text{ (or) } a + b$$

$$\oplus \text{ (or) } \vee \rightarrow \text{join}$$

Eg: Let $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$, $D = \{(a,b) / a|b\}$

Then the relation $D = \{(1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (1,24), (2,4), (2,6), (2,8), (2,12), (2,24), (3,6), (3,12), (3,24), (4,8), (4,12), (4,24), (6,12), (6,24), (8,24), (12,24)\}$

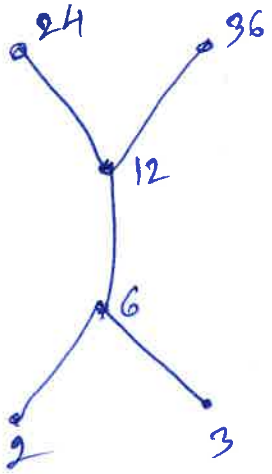


$[S_{24}, D]$ is lattice.

Example - ①

Determine the Posets given by the Hasse diagram are lattice.

(i)

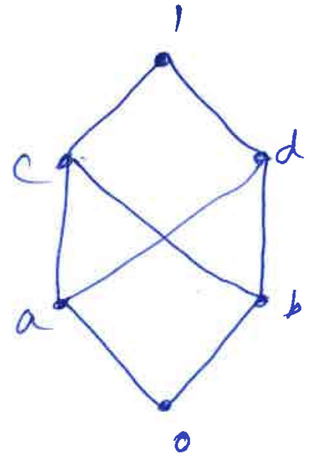


is not lattice

Since, LUB $\{24, 36\}$ does not Exists

& GLB $\{2, 3\}$ does not Exists

(ii)



is not Lattice

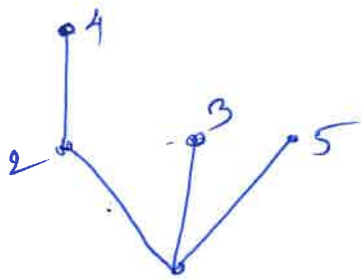
Since, LUB $\{a, b\}$ does not Exists

& GLB $\{c, d\}$ does not Exists

Example - ② Determine whether the Posets $\{(1, 2, 3, 4, 5), /\}$ are lattices.

Soln: Given that $\{(1, 2, 3, 4, 5), /\}$. Then the relation $D = \{(1,2), (1,3), (1,4), (1,5), (2,4)\}$

The Hasse diagrams



Here, LUB $\{2, 3\}$ = does not Exists

∴ The Poset $\{(1, 2, 3, 4, 5), /\}$ is not lattice.

Chapter - 5.2

Properties of lattice

1) **Idempotent law:-**

(i) $a \wedge a = a$, (ii) $a \vee a = a$

2) **Commutative law:**

(i) $a \wedge b = b \wedge a$, (ii) $a \vee b = b \vee a$

3) **Associative law:-**

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(ii) $(a \vee b) \vee c = a \vee (b \vee c)$

4) **Absorption law:-**

(i) $a \wedge (a \vee b) = a$; (ii) $a \vee (a \wedge b) = a$

Theorem:-①

Every finite lattice is bounded.

Proof:

Let $(L, \wedge, \vee)$ be a given lattice

Since L is a lattice both GLB and LUB exists

Let 'a' be GLB of L and 'b' be LUB of L.

then for any $x \in L$.

$$a \leq x \leq b \longrightarrow ①$$

$$GLB\{a, x\} = a \wedge x = a$$

$$LUB\{a, x\} = a \vee x = x$$

and $GLB\{x, b\} = x \wedge b = x$
$LUB\{x, b\} = x \vee b = b$

* Ho finite lattice have GLB & LUB

$\therefore$ Any finite lattice is bounded.

Theorem - ②
State and Prove Isotonicity Property of lattice.

Statement:
Let $(L, \vee, \wedge)$ be given lattice for any $a, b, c \in L$.
We have $b \leq c \Rightarrow$ (i) $a \wedge b \leq a \wedge c$
(ii) $a \vee b \leq a \vee c$

Proof:
Given that $b \leq c$
i.e, $GLB\{b, c\} = b \wedge c = b \longrightarrow ①$
$LUB\{b, c\} = b \vee c = c \longrightarrow ②$

Claim:- ① $a \wedge b \leq a \wedge c$

To Prove this, we want to Prove
$(a \wedge b) \wedge (a \wedge c) = a \wedge b$

L.H.S $(a \wedge b) \wedge (a \wedge c)$.
$\Rightarrow a \wedge [b \wedge (a \wedge c)]$ {Associative law}
$\Rightarrow a \wedge [(b \wedge a) \wedge c]$
$\Rightarrow a \wedge [(a \wedge b) \wedge c]$ {Commutative law}
$\Rightarrow (a \wedge a) \wedge (b \wedge c)$

Theorem-(3)

Let $(L, \wedge, \vee)$ be a lattice in which $\wedge$ and $\vee$ denote the operation of meet and join respectively for any $a, b \in L$, $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$

(or)

Let $(L, \wedge, \vee)$ be a lattice, then $a, b \in L$.

(i) $a \vee b = b$, iff $a \leq b$

(ii) $a \wedge b = a$, iff $a \leq b$

(iii) $a \wedge b = a$ iff $a \vee b = b$

Proof: (i) $\Rightarrow$ (ii)

Let $a \leq b$, form the definition of $a \vee b$,

We've $a \vee b \leq b \longrightarrow (1)$

Since, $a \vee b$ is the LUB $\{a, b\}$,

we've $b \leq a \vee b \longrightarrow (2)$

from (1) & (2) $\boxed{a \vee b = b}$

(ii) $\Rightarrow$ (iii)

Let $a \vee b = b \longrightarrow (3)$

Now, $a \wedge b = a \wedge (a \vee b)$ {using Eqn (3)}

$= a$ {by Absorption law}

$\boxed{a \wedge b = a}$

(iii) $\Rightarrow$ (i) Let $a \wedge b = a$

then the lower bound of $\{a, b\} = a$

$\Rightarrow \boxed{a \leq b}$

$\Rightarrow (a \wedge a) \wedge (b \wedge c)$ $\{a \wedge a = a\}$

$\Rightarrow a \wedge (b \wedge c)$ $\{$ using Equ ① $\}$

$\Rightarrow a \wedge b$

i.e., $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

$\therefore \boxed{a \wedge b \leq a \wedge c}$

Claim :- ② $a \vee b \leq a \vee c$

To Prove this, we want to Prove

$(a \vee b) \vee (a \vee c) = a \vee c$

L.H.S

$(a \vee b) \vee (a \vee c)$

$\Rightarrow a \vee [b \vee (a \vee c)]$ $\{$ Associative law $\}$

$\Rightarrow a \vee [(b \vee a) \vee c]$

$\Rightarrow a \vee [(a \vee b) \vee c]$ $\{$ Commutative law $\}$

$\Rightarrow (a \vee a) \vee (b \vee c)$ $\{a \vee a = a$

$\Rightarrow a \vee (b \vee c)$ $\{$ using Equ ② $\}$

$\Rightarrow a \vee c$

i.e., $(a \vee b) \vee (a \vee c) = a \vee c$

$\therefore \boxed{a \wedge b \leq a \wedge c}$

Theorem - (4)

In a lattice if $a \le b \le c$, then show that

i) $a \vee b = b \wedge c$

ii) $(a \wedge b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee b) = b$.

Proof: (i) Given that $a \le b \le c$

Since, $a \le b \Rightarrow \left.\begin{array}{l} a \vee b = b \\ \& \ a \wedge b = a \end{array}\right\} \rightarrow (1)$

$b \le c \Rightarrow \left.\begin{array}{l} b \vee c = c \\ \& \ b \wedge c = b \end{array}\right\} \rightarrow (2)$

$a \le c \Rightarrow \left.\begin{array}{l} a \vee c = c \\ \& \ a \wedge c = a \end{array}\right\} \rightarrow (3)$

from (1) & (2), we get

$a \vee b = b = b \wedge c$

$\boxed{a \vee b = b \wedge c}$

ii) $(a \wedge b) \vee (b \wedge c) = (a \vee b) \wedge (a \wedge c) = b$

L-H-S $(a \wedge b) \vee (b \wedge c)$

$\Rightarrow a \vee b$ {using (1) & (2)}

$\Rightarrow b \rightarrow (4)$

R-H-S

$(a \vee b) \wedge (a \wedge c)$

$\Rightarrow b \wedge c$ {using (1) & (2)

$\Rightarrow b \rightarrow (5)$

from (4) & (5), we get

$(a \wedge b) \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = b$.

Theorem-5 Modular Inequality

If $(L, \wedge, \vee)$ is a lattice, then for any $a, b, c \in L$

$a \leq c \iff a \vee (b \wedge c) \leq (a \vee b) \wedge c$

(or)

If $(L, \wedge, \vee)$ is a lattice, then for any $a, b, c \in L$

$a \leq c$ if and only if $a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Proof:

Let us assume, $a \leq c$

i.e, $a \vee c = c \longrightarrow (1)$

By distributive inequality, we've

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ {Using Eqn (1)}

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$

$\therefore a \leq c \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c \longrightarrow (2)$

Converse Part:

Let us assume.

$a \vee (b \wedge c) \leq (a \vee b) \wedge c$

By definition of LUB and GLB, we've.

$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$

$\Rightarrow a \leq c$

$\therefore a \vee (b \wedge c) \leq (a \vee b) \wedge c \Rightarrow a \leq c \longrightarrow (3)$

from (2) & (3), we get

$a \leq c \iff a \vee (b \wedge c) \leq (a \vee b) \wedge c.$

Defn: Distributive lattice

A lattice $(L, \wedge, \vee)$ is said to be distributive lattice if $\wedge$ and $\vee$ satisfies the following condition

D_1: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

D_2: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Theorem-(6)

Prove that any chain is a distributive lattice.

Proof:

Let $(L, \wedge, \vee)$ be a given chain, $\forall a, b \in L$.

Since, any 2-elements of a chain are comparable, we have $a \leq b$ (or) $b \leq a$

Case-(i) $a \leq b$

then $GLB\{a, b\} = a$

& $LUB\{a, b\} = b$

Case-(ii) $b \leq a$

then $GLB\{b, a\} = b$

& $LUB\{b, a\} = a$

In both cases, any 2-elements of a chain has both GLB and LUB

$\therefore$ Any chain is a lattice

Next, we want to prove $(L, \wedge, \vee)$ satisfies the distributive law.

Let $a, b, c \in L$, since any chain satisfies the Comparable Property, we have

<u>Case-1</u> :- $a \leq b \leq c$

<u>Case-2</u> :- $a \leq c \leq b$

<u>Case-3</u> :- $b \leq a \leq c$

<u>Case-4</u> :- $b \leq c \leq a$

<u>Case-5</u> :- $c \leq a \leq b$

<u>Case-6</u> :- $c \leq b \leq a$

<u>Case-1</u> . $\boxed{a \leq b \leq c}$

<u>To Prove</u>: D_1 : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

<u>L.H.S</u> $a \vee (b \wedge c)$

$\Rightarrow a \vee b$ $\{b \leq c \Rightarrow b \wedge c = b$

$\Rightarrow b$ $\{a \leq b \Rightarrow a \vee b = b$

<u>R.H.S</u> $(a \vee b) \wedge (a \vee c)$

$\Rightarrow b \wedge c$ $\begin{cases} a \leq b \Rightarrow a \vee b = b \\ a \leq c \Rightarrow a \vee c = c \end{cases}$

$\Rightarrow b$ $\{b \leq c \Rightarrow b \wedge c = b$

$\therefore$ L.H.S = R.H.S

D_1 Condition is true for Case-1

III ly we have remaining 5 cases.

$\therefore (L, \wedge, \vee)$ is a distributive lattice

$\therefore$ Any chain is a distributive lattice.

Theorem - 7

State and Prove distributive inequality lattice

Statement:

Let $(L, \wedge, \vee)$ be a given lattice. for any $a, b, c \in L$ then following inequality holds

i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Proof:

Claim - 1 $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

from the definition of LUB, it is obvious that

$$a \leq a \vee b \longrightarrow (1)$$

and $b \wedge c \leq b \leq a \vee b$

$$\Rightarrow b \wedge c \leq a \vee b \longrightarrow (2)$$

from (1) & (2), $a \vee b$ is upper bound of $\{a, b \wedge c\}$

Hence, $a \vee b \geq a \vee (b \wedge c) \longrightarrow (3)$

from the definition of LUB, it is obvious that

$$a \leq (a \vee c) \longrightarrow (4)$$

and $b \wedge c \leq c \leq a \vee c$

$$\Rightarrow b \wedge c \leq a \vee c \longrightarrow (5)$$

from (4) & (5), $a \vee c$ is upper bound $\{a, b \wedge c\}$

Hence, $a \vee c \geq a \vee (b \wedge c) \longrightarrow (6)$

from (3) & (6)

we have $a \vee (b \wedge c)$ is a lower bound ~~of~~

~~of~~ $\{a \vee b, a \vee c\}$

$\therefore \; a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Claim - ②

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

W.K.T $a \geq a \wedge b \longrightarrow ①$

and $b \wedge c \geq b \geq a \wedge b$

$b \wedge c \geq a \wedge b \longrightarrow ②$

from ① & ②, $a \wedge b$ is lower bound of $\{a, b \vee c\}$

Hence, $a \wedge b \leq a \wedge (b \vee c) \longrightarrow ③$

W.K.T $a \geq a \wedge c \longrightarrow ④$

and $b \vee c \geq c \geq a \wedge c$

$b \vee c \geq a \wedge c \longrightarrow ⑤$

from ④ & ⑤, $a \wedge c$ is lower bound of $\{a, b \vee c\}$

Hence, $a \wedge c \leq a \wedge (b \vee c) \longrightarrow ⑥$

from ③ & ⑥, we have

$a \wedge (b \vee c)$ is an upper bound of $\{a \wedge b, a \wedge c\}$

$\therefore \; a \wedge (b \wedge c) \geq (a \wedge b) \vee (a \wedge c)$.

Eg - ① In any distributive lattice $(L, \wedge, \vee)$, $\forall\, a, b, c \in L$

Prove that $a \vee b = a \vee c$ & $a \wedge b = a \wedge c \Rightarrow b = c$.

Soln: Given that $a \vee b = a \vee c$ & $a \wedge b = a \wedge c$

Consider, $b = b \vee (b \wedge a)$ Absorption law

$= b \vee (a \wedge b)$ Commutative law

$= b \vee (a \wedge c)$ Given Condition

$= (b \vee a) \wedge (b \vee c)$ D_1 - Condition

$= (a \vee b) \wedge (b \vee c)$ Commutative law

$= (a \vee c) \wedge (b \vee c)$ Given condition

$= (c \vee a) \wedge (c \vee b)$ Commutative law

$= c \vee (a \wedge b)$ D_1 - Condition

$= c \vee (a \wedge c)$ Given Condition

$= c \vee (c \wedge a)$ Commutative law

$= c$ Absorption law

$\therefore\ b = c$.

Eg - (2) Show that Every totally ordered set is a lattice.

Soln: Let $(L, \wedge, \vee)$ be a given totally ordered set (chain)

and $\forall\ a, b \in L$.

Since, any 2-Elements of a totally ordered set

are comparable,

we have Either $a \leq b$ (or) $b \leq a$

Case - (i) $a \leq b$	Case - (ii) $b \leq a$
then GLB $\{a, b\} = a$	then GLB $\{a, b\} = b$
& LUB $\{a, b\} = b$	& LUB $\{a, b\} = a$

In both cases, any 2-Elements of a totally ordered set has both GLB and LUB

$\therefore$ Any totally ordered set is a lattice

Eg - (3) In any distributive lattice $(L, \wedge, \vee)$. Prove that

$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$.

Soln: Given that $(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

L.H.S $(a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$\Rightarrow (a \vee b) \wedge [(c \vee b) \wedge (c \vee a)]$ Commutative law

$\Rightarrow (a \vee b) \wedge [c \vee (b \wedge a)]$ distributive law

$\Rightarrow (a \vee b) \wedge [c \vee (b \wedge a)]$

$\Rightarrow \{(a \vee b) \wedge c] \vee [(a \vee b) \wedge (b \wedge a)]$ distributive law

$\Rightarrow [(a \vee b) \wedge c] \vee [(a \vee b) \wedge (a \wedge b)]$ Commutative law

$\Rightarrow [c \wedge (a \vee b)] \vee [(a \wedge b) \wedge (a \vee b)]$ Commutative law

$\Rightarrow [(c \wedge a) \vee (c \wedge b)] \vee [(a \wedge b) \wedge a] \vee [(a \wedge b) \wedge b]$

$\Rightarrow (c \wedge a) \vee (c \wedge b) \vee (a \wedge b) \vee (a \wedge b)$ { Idempotent law

$\Rightarrow (c \wedge a) \vee (c \wedge b) \vee (a \wedge b)$

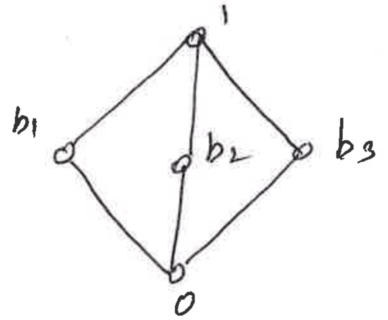
$\Rightarrow (c \wedge a) \vee (b \wedge c) \vee (a \wedge b)$ { Commutative law

$\Rightarrow (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

$\Rightarrow$ R.H.S

$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Eg - 4 Examine whether the lattice given in the following Hasse diagram is distributive (or) not.



Soln:

To check the given lattice is distributive, we have to check the following distributive law

D_1 : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $\forall\ a, b, c \in L$.

Consider the (b_1, b_2, b_3)

Now, $L.H.S = a \vee (b \wedge c) = b_1 \vee (b_2 \wedge b_3)$

$= b_1 \vee 0$

$= b_1 \longrightarrow ①$

$R.H.S = (a \vee b) \wedge (a \vee c)$

$= (b_1 \vee b_2) \wedge (b_1 \vee b_3)$

$= 1 \wedge 1$

$= 1 \longrightarrow ②$

from ① & ②

$L.H.S \neq R.H.S$

$b_1 \neq 1$

$\therefore a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

for $\{b_1, b_2, b_3\}$ not satisfied the distributive law

$\therefore$ The given lattice is not distributive.

Defn Sublattice

Let $(L, \wedge, \vee)$ be a lattice and $S \subset L$ be a subset of L. The algebra $(S, \wedge, \vee)$ is a sublattice of $(L, \wedge, \vee)$ iff S is closed under both operations $\wedge$ and $\vee$.

Defn Lattice Homomorphism

Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices. A mapping $g : L \to S$ is called a lattice homomorphism from the lattice $(L, *, \oplus)$ to $(S, \wedge, \vee)$ if for any $a, b \in L$

$$g(a * b) = g(a) \wedge g(b) \text{ and } g(a \oplus b) = g(a) \vee g(b).$$

Defn Bounded lattice

A lattice $(L, \leq)$ is said to be a bounded lattice it has both 0 (min) and 1 (max) Elements denoted by $(L, \leq, 0, 1)$.

Defn Complement

Let $(L, \leq, 0, 1)$ be a bounded lattice we say that Complement at $a = a'$ (or) $\bar{a} = b$ if $a \wedge b = 0$ and $a \vee b = 1$. (ie) $a \wedge a' = 0$ and $a \vee a' = 1$

Defn Complement lattice

Let $(L, \leq, 0, 1)$ be a bounded lattice. Every Element of L has Complement then it is Called Complement lattice $(a \wedge a' = 0, a \vee a' = 1)$.

Eg - ①

In a complemented distributive lattice, show that the following are Equivalent.

$$a \le b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \le a'$$

(OR)

In otherwords the following are Equivalent

(i) $a \le b$ (ii) $a \wedge b' = 0$, (iii) $a' \vee b = 1$, (iv) $b' \le a'$

Soln:

Given that, (i) $a \le b$, (ii) $a \wedge b' = 0$, (iii) $a' \vee b = 1$, (iv) $b' \le a'$

(i) $\Rightarrow$ (ii)

Let $a \le b$, then $a \wedge b = a$ & $a \vee b = b \longrightarrow ①$

then $a \wedge b' = [(a \wedge b) \wedge b']$ Using Eqn ①

$= a \wedge (b \wedge b')$ Associative law

$= a \wedge 0$ $\{ b \wedge b' = 0$

$a \wedge b' = 0$

Hence, $a \le b \Rightarrow a \wedge b' = 0$

(ii) $\Rightarrow$ (iii)

Let $a \wedge b' = 0$

then, taking complement on both side

$(a \wedge b')' = 0'$

$a' \vee (b')' = 1$

$a' \vee b = 1$

Hence, $a \wedge b' = 0 \Rightarrow a' \vee b = 1$

(iii) $\Rightarrow$ (iv)

Let $a' \vee b = 1$, then

$\Rightarrow (a' \vee b) \wedge b' = 1 \wedge b'$ {Cancellation law}

$\Rightarrow (a' \wedge b') \vee (b \wedge b') = b'$ {distributive law}

$\Rightarrow (a' \wedge b') \vee 0 = b'$ {$b \wedge b' = 0$}

$\Rightarrow a' \wedge b' = b'$

$\Rightarrow b' \leq a'$

Hence, $a' \vee b = 1 \Rightarrow b' \leq a'$

(iv) $\Rightarrow$ (i)

Let $b' \leq a'$, then

$a' \wedge b' = b'$

Taking Complement on both sides.

$(a' \wedge b')' = (b')'$

$\Rightarrow (a')' \vee (b')' = b$

$\Rightarrow a \vee b = b$

$\Rightarrow a \leq b$

Hence, $b' \leq a' \Rightarrow a \leq b$.

Eg - ② Prove that in a complemented distributive lattice complement is unique.

(OR)

If $(L, \vee, \wedge, 0, 1)$ is a distributive lattice then each element $x \in L$ has atmost one complement.

Soln -

Let us assume x and y are two complement of a

Since, x is a complement of a, we've

$$a \wedge x = 0 \ \& \ a \vee x = 1 \longrightarrow ①$$

Since, y is a complement of a, we've

$$a \wedge y = 0, \ a \vee y = 1 \longrightarrow ②$$

Now,

$$\begin{aligned} x &= x \vee 0 \\ &= x \vee (a \wedge y) && \{\text{using Eqn ②}\} \\ &= (x \vee a) \wedge (x \vee y) && \{\text{distributive law}\} \\ &= (a \vee x) \wedge (x \vee y) && \{\text{Commutative law}\} \\ &= 1 \wedge (x \vee y) && \{\text{using Eqn ①}\} \end{aligned}$$

$$x = x \vee y \longrightarrow (A)$$

Again,

$$\begin{aligned} y &= y \vee 0 \\ &= y \vee (a \wedge x) && \{\text{using Eqn ①}\} \\ &= (y \vee a) \wedge (y \vee x) && \{\text{distributive law}\} \end{aligned}$$

$= (y \vee a) \wedge (y \vee x)$

$= (a \vee y) \wedge (y \vee x)$ {Commutative law}

$= 1 \wedge (y \vee x)$ {applying ①}

$y = y \vee x \longrightarrow$ (B)

from (A) & (B)

$x = x \vee y = y \vee x = y$

$\Rightarrow x = y$

$\therefore$ Complement is unique

Theorem: State and prove Demorgan's law of lattice

(OR)

If $(L, \wedge, \vee, 0, 1)$ is a complemented lattice, then prove that, (i) $(a \wedge b)' = a' \vee b'$ (or) $\overline{(a \wedge b)} = \bar{a} \vee \bar{b}$

(ii) $(a \vee b)' = a' \wedge b'$ (or) $\overline{(a \vee b)} = \bar{a} \wedge \bar{b}$

Proof: Claim-1 $(a \wedge b)' = a' \vee b'$

To prove the above, it is enough to prove that

(i) $(a \wedge b) \wedge (a' \vee b') = 0$

(ii) $(a \wedge b) \vee (a' \vee b') = 1$

(i) $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$

$\Rightarrow [a \wedge b \wedge a'] \vee [a \wedge b \wedge b']$

$\Rightarrow [a \wedge b \wedge a'] \vee [a \wedge b \wedge b']$

$\Rightarrow [(a \wedge a') \wedge b] \vee [a \wedge (b \wedge b')]$

$\Rightarrow [0 \wedge b] \vee [a \wedge 0]$

$\Rightarrow (0 \wedge b) \vee (0 \wedge a)$

$\Rightarrow 0 \vee 0$

$\Rightarrow 0$

$\therefore (a \wedge b) \wedge (a' \vee b') = 0 \longrightarrow$ (1)

$\left\{ \cancel{a \wedge a} \right.$

$\left\{ \begin{array}{l} a \wedge a' = b \wedge b' = 0 \\ 0 \wedge a = 0 \wedge b = 0 \end{array} \right.$

(ii) $(a \wedge b) \vee (a' \vee b')$

$\Rightarrow [a \vee (a' \vee b')] \wedge [b \vee (a' \vee b')]$

$\Rightarrow [a \vee a' \vee b'] \wedge [b \vee a' \vee b']$

$\Rightarrow [(a \vee a') \vee b'] \wedge [b \vee b' \vee a']$

$\Rightarrow [(a \vee a') \vee b'] \wedge [(b \vee b') \vee a']$

$\Rightarrow (1 \vee b') \wedge (1 \vee a')$

$\Rightarrow (b' \vee 1) \wedge (1 \vee a')$

$\Rightarrow 1 \wedge 1$

$\Rightarrow 1$

$\therefore (a \wedge b) \vee (a' \vee b') = 1 \longrightarrow$ (2)

$\left\{ \begin{array}{l} a \vee a' = b \vee b' = 1 \\ 1 \vee a = 1 \vee b = 1 \end{array} \right.$

from (1) & (2)

$(a \wedge b)' = a' \vee b'$

Claim-② $(a \vee b)' = a' \wedge b'$

It is enough to prove that.

(i) $(a \vee b) \wedge (a' \wedge b') = 0$

(ii) $(a \vee b) \vee (a' \wedge b') = 1$

(i) $(a \vee b) \wedge (a' \wedge b')$

$\Rightarrow [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]$

$\Rightarrow [a \wedge a' \wedge b'] \vee [b \wedge a' \wedge b']$

$\Rightarrow [(a \wedge a') \wedge b'] \vee [b \wedge b' \wedge a']$

$\Rightarrow [(a \wedge a') \wedge b'] \vee [(b \wedge b') \wedge a']$

$\Rightarrow [0 \wedge b') \vee (0 \wedge a')$

$\Rightarrow 0 \vee 0$

$\Rightarrow 0$

$\therefore (a \vee b) \wedge (a' \wedge b') = 0 \longrightarrow$ ③

(ii) $(a \vee b) \vee (a' \wedge b')$

$\Rightarrow [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$

$\Rightarrow [a \vee b \vee a'] \wedge [a \vee b \vee b']$

$\Rightarrow [(a \vee a') \vee b] \wedge [a \vee (b \vee b')]$

$\Rightarrow [1 \vee b] \wedge [a \vee 1]$

$\Rightarrow \{1 \vee b) \wedge (1 \vee a)$

$\Rightarrow 1 \wedge 1$

$\Rightarrow 1$

$\therefore (a \vee b) \vee (a' \wedge b') = 1 \longrightarrow$ ④

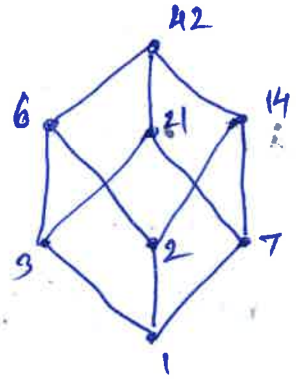
from ③ & ④

$(a \vee b)' = a' \wedge b'$

Eg: If S_{42} is the set all divisors of 42 and D is the relation "divisor of" on S_{42}, Prove that $\{S_{42}, D\}$ is a Complemented lattice.

Soln: Let $S_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$

The Hasse diagram $[S_{42}, D]$



Here, '0' Element (or) least Element is 1

1 Element (or) greatest Element is 42

$LCM\{1, 42\} = 1 \vee 42 = 42$

$GCD\{1, 42\} = 1 \wedge 42 = 1$

$\therefore$ Complement of 1 is 42.

ie, $1' = 42$

|||ly $2 \wedge 21 = 1$ & $2 \vee 21 = 42$

$2' = 21$

$3 \wedge 14 = 1$ & $3 \vee 14 = 42$

$3' = 14$

$6 \wedge 7 = 1$ & $6 \vee 7 = 42$

$6' = 7$

obvisously, $7' = 6$, $14' = 3$, $21' = 2$ & $42' = 1$

Since, Every Element of S_{42} has Complement.

$\therefore$ $[S_{42}, D]$ is a complemented lattice.

Chapter-5.4

BOOLEAN ALGEBRA

Defn Boolean Algebra

A Complemented distributive lattice is Called Boolean Algebra.

(or)

A Boolean algebra is distributive lattice with '0' Element and '1' Element in which Every Element has a Complement.

(or)

A Boolean algebra is a non-Empty set with 2-binary operations $\wedge$ and $\vee$ as satisfied by the following Conditions, $\forall a, b, c \in L$.

1) L_1: $a \wedge a = a$ & $a \vee a = a$

2) L_2: $a \wedge b = b \wedge a$ & $a \vee b = b \vee a$

3) L_3: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$. & $a \vee (b \vee c) = (a \vee b) \vee c$

4) L_4: $a \wedge (a \vee b) = a$ & $a \vee (a \wedge b) = a$

5) D_1: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

6) D_2: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

7) $\exists$ a Elements '0' and '1' such that

$a \wedge 0 = 0$, $a \vee 0 = a$, $a \wedge 1 = a$; $a \vee 1 = 1$. $\forall a$

8) $\forall a \in L$, $\exists$ Corresponding Element a' in L such that $a \wedge a' = 0$ and $a \vee a' = 1$.

Eg - ① If $a, b \in S$; where $S = \{1, 2, 3, 6\}$ and $a + b = lcm(a, b)$ & $a \cdot b = gcd(a, b)$ and $a' = b/a$. Show that $\{S, +, \cdot, 1, 6\}$ is a Boolean Algebra.

Soln:

Given that $S = \{1, 2, 3, 6\}$, $a, b \in S$ & $a' = b/a$

and $a + b = lcm(a, b)$ & $a \cdot b = gcd(a, b)$

Zero Element = 0 Element = 1

One Element = 1 Element = 6

If 'a' is any of the Element $\{1, 2, 3, 6\}$ of S,

then $a \wedge 0 = lcm(a, 1) = a$

$a \vee 1 = gcd(a, b) = a$

$\therefore$ Identity law holds

|||ly we can verify associative & distributive law

Now, $a \wedge a' = lcm(a, b/a) = 6$ (1 Element)

$a \vee a' = gcd(a, b/a) = 1$ (0 Element)

$\therefore$ The Complement law also holds

Hence, $(S, +, \cdot, 1, 6)$ is a Boolean Algebra.

Eg - ② If $P(S)$ is the Power set of a non-Empty set S, Prove that $\{P(S), \cup, \cap, ', \phi, S\}$ is Boolean Algebra.

Soln

Given that $P(S)$ = Power set is 2^n $\begin{cases} 2^2 = 4 \\ 2^3 = 8 \end{cases}$

It is noticed that for any 2-subset A and B of $P(S)$.

(i) $P(S)$ is a lattice

let $A, B \in P(S)$, then clearly $A \cup B = \text{lub}\{A, B\} \in P(S)$

& $A \cap B = \text{gcd}\{A, B\} \in P(S)$

$\therefore [P(S), \subseteq]$ is a lattice

(ii) Moreover $P(S)$ is a distributive lattice

$D_1: A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$D_2: A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iii) Here, '0' element is Empty set ϕ

'1' element is given set S

Complement of $a = a' = S - a$

Since, $\{P(S), \cup, \cap\}$ is a Complemented distributive lattice

$\therefore \{P(S), \cup, \cap, ', \phi, S\}$ is an Boolean Algebra

Eg-(3) In any Boolean Algebra, show that

$$(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$$

Soln Given that

$$(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$$

L.H.S

$$(a+b')(b+c')(c+a')$$

$\{a+0=a\}$
$\{aa'=0\}$

$$= (a+b'+0)(b+c'+0)(c+a'+0)$$

$$= (a+b'+cc')(b+c'+aa')(c+a'+bb')$$

$$= (a+b'+c)(a+b'+c')(b+c'+a)(b+c'+a')(c+a'+b)(c+a'+b')$$

$$= [(a'+b+c)(a'+b+c')][(b'+c+a)(b'+c+a')][(c'+a+b)(c'+a+b')]$$

$$= (a'+b+cc')(b'+c+aa')(c'+a+bb')$$

$$= (a'+b+0)(b'+c+0)(c'+a+0)$$

$$= (a'+b)(b'+c)(c'+a)$$

$$= R.H.S$$

$$\therefore (a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$$